I YEAR – I SEMESTER
COURSE CODE: 7MMA1C3

CORE COURSE-III – DIFFERENTIAL GEOMETRY

Unit I

Space Curves – Definition of a space Curve – Arc length – tangent – normal and binormal – Curvature and Torsion – Contact between Curves and Surfaces – tangent surface – Involutes and evolutes – Intrinsic equations – Fundamental Existence Theorem for space Curves - Helices.

Unit II

Intrinsic Properties of a Surface – Definition of a Surface – Curves on a Surface – Surface of revolution – Helicoids – Metric – Direction Coefficients – families of Curves – Isometric Correspondence – Intrinsic properties.

Unit III

Geodesics – Canonical geodesic equations – Normal property of geodesics – Existence Theorems – Geodesic parallels.

Unit IV

Geodesic Curvature – Gaurs – Bonnet Theorem – Gaussian Curvature – Surface of Constant Curvature.

Unit V

Non-Intrinsic Properties of a Surface – The second fundamental form – Principal Curvature – Lines of Curvature – Developable – Developable associated with space curves and with curves on surfaces.

Text Book

T.J.Willmore, An Introduction to Differential Geometry, Oxford University Press (17^{th} Impression) New Delhi 2002 (Indian Print)

Chapter I	:	Sections 1 to 9
Chapter II	:	Sections 1 to 9
Chapter II	:	Sections 10 to 14
Chapter II	:	Sections 15 to 18
Chapter III	:	Sections 1 to 6

Books for Supplementary Reading and Reference:

1. D.Somasundaram, Differential Geometry, A First Course, Narosa Publishing House, Chennai, 2005.
2. D.J.Struik, Classical Differential Geometry, Addison Wesley Publishing Company INC, Massachusetts, 1961.

Unit :- 1

Introductory Remarks about space curves:

Definitions :- PARAMETER.

A curve is the Locus of the Point whose position vector $\vec{r}$ with respect to the fixed origin is a function of a single variable u is known as Parameter.

A curve in a plane can be Given in the parametric form by the eqns. $x = x(u), y = y(u)$.

Eg:
The circle $x^2+y^2=a^2$ with centre at origin and radius a has a parametric form $x = a\cos u$, $y = a\sin u$

Definitions:- Surface

A surface is defined as the Locus of a Point whose Cartesian co-ordinates (x,y,z) are the functions of two independent Parameters (u,v) then we have $x = f(u,v)$ $y = g(u,v)$ $z = h(u,v)$.

Definitions

A curve in space Given by the eqn $x = x(u), y = y(u), z = z(u)$ where [illegible] can be represented in the parametric form.

Eg:-

The sphere $x^2+y^2+z^2=a^2$ with centre at origin and radius a can be written in the parametric form

$x = a\sin u\cos v,\ y = a\sin u\sin v,\ z = a\cos u$

Note:-

1) $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$

$= x(u)\vec{i}+y(u)\vec{j}+z(u)\vec{k}$

$= \vec{R}(u)$

2) A space curve can be expressed as the intersection of two surfaces

$f(x,y,z)=0 \rightarrow (1)$

$g(x,y,z)=0 \rightarrow (2)$

The parametric form of the ~~form~~ space curve is $x=x(u),\ y=y(u),\ z=z(u)$. $\rightarrow (3)$

The eqn (3) can be transformed into eqn (1) & (2) by eliminating u from (3).

Problem:-

Find the eqn of the curve whose parametric eqns are $x=u,\ y=u^2,\ z=u^3$

Soln:

Let us take $xy = u\cdot u^2 = u^3 = z$

$\Rightarrow xy = z$

Also we take $xz = u\cdot u^3 = u^2\cdot u^2 = y^2$

$\Rightarrow xz = y^2$.

2M

Definitions:

1 FUNCTION OF CLASS m.

Let I be real interval and m be the positive integer. A real valued function definition on I is said to be the class of m if

i) f has mth derivative at every point of I

ii) Each derivative is continous on I.

Definition:

Regular function:

A Function R is said to be a regular function if the derivative $\frac{dR}{du} = R' \neq 0$ on the real interval I

PATH OF CLASS m:

A regular vector valued function of class m is called path of class m

EQUIVALENT PATH:

Two paths R_1 & R_2 of same class m on the intervals I_1, I_2 are said to be equivalent if there exists a strictly increasing function ϕ of class m which map I_1 onto I_2 such that $R_1 = R_2 \circ \phi$.

3/7/14. Arc length.

Derive an expression for the arc length of a curve in space of the form $s = s(u) = \int_{u_0}^{u} |\dot{R}(u)|\, du$

Proof:-

Let $\vec{r} = \vec{R}(u)$ be a path and a, b be two real numbers where $a < u < b$.

distance.

Take any subdivision Δ of the interval $[a, b]$ by the points $a = u_0 < u_1 < u_2 \ldots < u_n = b$ then the corresponding length

$$L\Delta = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})|$$

$$= \sum_{i=1}^{n} \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\, du \right|$$

$$\leq \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} |\dot{R}(u)|\, du$$

$$= \sum_{i=1}^{n} \int_{a=u_0}^{b=u_n} |\dot{R}(u)|\, du$$

$$\therefore L\Delta \leq \int_a^b |\dot{R}(u)|\, du$$

then $S = s(u)$ denote the arc length from a to any point then the arc length from u_0 to u is

$$s(u) - s(u_0) \le \int_{u_0}^{u} |R'(u)|\,du \rightarrow (1)$$

then by the definition of arc length we have,

arc length

$$|R(u) - R(u_0)| \le s(u) - s(u_0). \rightarrow (2)$$

$\div$ by $|u - u_0|$

$$\left|\frac{R(u) - R(u_0)}{u - u_0}\right| \le \frac{s(u) - s(u_0)}{u - u_0}$$

$$\le \frac{1}{u - u_0} \int_{u_0}^{u} |R'(u)|\,du \quad [\because \text{by eqn } (1)].$$

$\frac{f(x+h) - f(x)}{h} = f'(x)$

$$\Rightarrow \lim_{u \to u_0} \left|\frac{R(u) - R(u_0)}{u - u_0}\right| \le \lim_{u \to u_0} \left[\frac{1}{u - u_0} \int_{u_0}^{u} |R'(u)|\,du\right]$$

$$\Rightarrow |R'(u_0)| \le \dot{s}(u) \le |R'(u_0)|$$

$\Rightarrow \dot{s}(u) = |R'(u_0)|$, since this is true for any u_0 in the range I of the parameter, it follows that s is the function of same class and that

$$s(u) = \int_{u_0}^{u} |R'(u)|\,du.$$

Corollary:

Note: Expression of arc length in cartesian parametric representation is

$$s = \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du.$$

Pf

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

ie) $\vec{r} = (x, y, z)$

then $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$

We know that.

$$\dot{\vec{r}} = \dot{R}(u)$$

$$\therefore |\dot{\vec{r}}| = |\dot{R}(u)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

We know that

$$s = s(u) = \int_a^u |\dot{R}(u)|\, du$$

$$= \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du$$

Note:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Prove that

i) $\vec{r} = (a\cos u, a\sin u, bu)$

$0 \le u \le \infty$

ii) $\vec{r} = \left(a\left(\frac{1 - v^2}{1 + v^2}\right), \frac{2av}{1 + v^2}, 2b\tan^{-1} v\right)$

$0 \le v \ldots$

are equivalent representation for ...

circular Helix.

Proof:- Given that,

$\vec{r} = (a\cos u,\ a\sin u,\ bu)$, $0 \le u \le \infty$ $\longrightarrow ①$

$\vec{r} = \left(a\left(\frac{1-v^2}{1+v^2}\right),\ \frac{2av}{1+v^2},\ 2b\tan^{-1}v\right)$, $\longrightarrow ②$

$0 \le v \le \infty$

To prove that, eqns ① & ② are equivalent.

It is nothing to prove that

(i) $a\left(\frac{1-v^2}{1+v^2}\right) = a\cos u$

(ii) $\frac{2av}{1+v^2} = a\sin u$

(iii) $2b\tan^{-1}v = bu$

for (i)

Let $v = \tan u/2$ sub in (i)

$a\left(\frac{1-v^2}{1+v^2}\right) = a\left(\frac{1-\tan^2 u/2}{1+\tan^2 u/2}\right)$

$= a\cos 2 \cdot u/2$

$= a\cos u$

$\left[\because \cos 2A = \frac{1-\tan^2 u/2}{1+\tan^2 u/2}\right.$

for (ii). Sub $v = \tan u/2$.

$$\frac{2av}{1+v^2} = \frac{2a\tan u/2}{1+\tan^2 u/2}$$

$$= \left(\frac{2\tan u/2}{1+\tan^2 u/2}\right) a$$

$$= a\sin 2(u/2)$$

$$= a\sin u \quad \left[\because \sin 2A = \frac{2\tan A}{1+\tan^2 A}\right]$$

for (iii).

Sub. $v = \tan u/2$ in (iii).

$$2b\tan^{-1} v = 2b\tan^{-1}(\tan u/2)$$

$$= 2b\, u/2$$

$$= bu$$

$\therefore$ eqn ① & ② are equivalent representation of circular Helix

Hence it's Proved.

Problem:

Obtain the equation of the circular Helix.

$\vec{r} = (a\cos u, a\sin u, bu)$,

$-\infty \le u \le \infty$

Where $a > 0$, refered to s as parameter, and show that the length of the one complete turn of Helix is $2\pi c$, where $c = \sqrt{a^2+b^2}$ (or).

find the length of the circular Helix $\vec{r}(u) = a\cos(u)\vec{i} + a\sin(u)\vec{j} + bu$ from $(0,0,0)$ to $(0,0,2\pi c)$ also obtain its equation interms of Parameter s. Then find the length of one complete turn of circular Helex.

$\vec{r} = a\cos u\,\vec{i} + a\sin u\,\vec{j} + bu\,\vec{k}$, $-\infty < u < \infty$

$(a\cos u,\ a\sin u,\ bu)$

soln:-

Given $\vec{r} = (a\cos u,\ a\sin u,\ bu) \longrightarrow (1)$

$\vec{r} = a\cos u\,\vec{i} + a\sin u\,\vec{j} + bu\,\vec{k}$

$\dot{\vec{r}} = -a\sin u\,\vec{i} + a\cos u\,\vec{j} + b\vec{k}$

$x : a\sin u$

$y : a\cos u$

$z : b$

$|\dot{\vec{r}}| = \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2}$

$= \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2}$ $\left[s = \int_a \sqrt{x^2+y^2+z^2} \right.$

$\int_0^a c\,du$

$= \sqrt{a^2(1) + b^2}$

$= \sqrt{a^2+b^2}$

We know that, $s = \int_a^u |R'(u)|du = \int_a^u \dot{\vec{r}}\,du$

$= \int_a^u \sqrt{a^2+b^2}\,du$

No Question

Formula: $\boxed{\cosh^2\theta - \sinh^2\theta = 1}$

Question $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = a\cosh(z/a)$

$\therefore z/a = u$

$x = a\cosh u$

$y = b\sinh u$

$z = au$

$$s = \int_0^u |\vec{r}'(u)|\,du$$

Given $\vec{r} = a\cosh u\,\vec{i} + b\sinh u\,\vec{j} + au\,\vec{k}$

$$\vec{r}' = a\sinh u\,\vec{i} + b\cosh u\,\vec{j} + a\vec{k}$$

$$|\vec{r}'| = \sqrt{a^2\sinh^2 u + b^2\cosh^2 u + a^2}$$

$$= \sqrt{a^2\sinh^2 u + b^2(1+\sinh^2 x) + a^2}$$

$$= \sqrt{a^2\sinh^2 u + b^2 + b^2\sinh^2 u + a^2}$$

$$= \sqrt{(a^2+b^2)\sinh^2 u + (a^2+b^2)}$$

$$= \sqrt{(a^2+b^2)(1+\sinh^2 u)}$$

$$= \sqrt{(a^2+b^2)\cosh^2 u}$$

$$= \sqrt{a^2+b^2}\,\cosh u$$

$$\therefore s = \int_0^u \sqrt{a^2+b^2}\,\cosh u\,du$$

$$= \sqrt{a^2+b^2}\,(\sinh u)_0^u$$

$$= \sqrt{a^2+b^2}\,.\,\sinh u$$

$$= \sqrt{a^2+b^2}\;y/b \quad \left[\text{from } y = (\sinh u)\,b \Rightarrow y/b = \sinh u\right]$$

Tangent, Normal, Binormal.

Defn:-

Tangent Line.

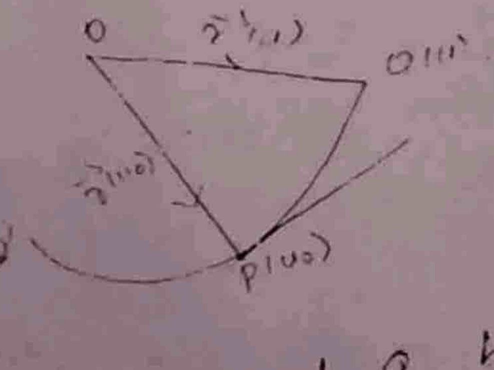
The tangent line to the curve at the point $P(u_0)$ on C is defined as the limiting position of a straight line "L" through the point $P(u_0)$ and the neighbouring point $Q(u)$ where $Q \to P$ along the curve C.

Theorem:

Find the unit tangent to a curve.

Pf:

Let γ be any curve represented by $\vec{r} = \vec{r}(u_0)$ and also γ be of class greater than or equal to one.



Let P and Q be two neighbouring points on the curve γ then represented by the parameters u_0 and u resp.

Let $\overline{OP} = \vec{r}(u_0)$

$\overline{OQ} = \vec{r}(u)$.

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP}$

$= \vec{r}(u) - \vec{r}(u_0)$

Since r be of class greater than or equal to one.

By Taylor's thm, we have

$\vec{r}(u) = \vec{r}(u_0) + (u-u_0)\vec{r}\,'(u_0) + (u-u_0)^2 \vec{r}\,''(u_0) + O(u-u_0)^3$

Then we have as $u \to u_0$.

$$\lim_{u \to u_0} \frac{r(u) - r(u_0)}{|r(u) - r(u_0)|}$$

When $Q \to P$

$$= \frac{\displaystyle \lim_{u \to u_0} \frac{r(u) - r(u_0)}{u - u_0}}{\displaystyle \lim_{u \to u_0} \left| \frac{r(u) - r(u_0)}{u - u_0} \right|}$$

$$\frac{\dot{\vec{r}}(u)}{|\dot{\vec{r}}(u)|}$$ This is called the unit tangent vector or it is denoted by t

Note:

$$\boxed{\vec{t} = \frac{\dot{\vec{r}}}{\dot{s}}}$$

$s = \int |\dot{\vec{r}}| \, ds$

$\Rightarrow \dot{s} = |\dot{\vec{r}}|$

$$= \frac{dr/du}{ds/du}$$

$$= dr/ds$$

' → das

· → dot

unit tagent vector ← $\vec{t} = \vec{r}'$

Osculating plane (or) plane of curvature.

Defn/-

2M 1) Let ∂ be a class of greaterthan or equal to 2.

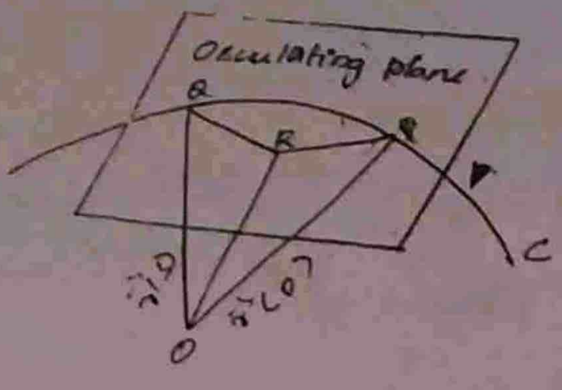
consider the two neighbouring point P and Q ... then the ~~oscillat~~ osculating point P and Q.

If the limiting position of the plane which contains tangent at P and Q parallel ~~to~~ Q.

2) Let C be a curve of class greater than or equal to 2. consider the two neighbouring point P and Q then the osculating point P and Q of the limiting position of the plane which contain tangent at P and $\parallel^{el}$ to tangent at Q.

Thm:

Find the equation of the Osculating plane.



Pf:

Let γ be a curve of class ≥ 2.

Let P and Q be two neighbouring points on the curve γ represented by the parameters 0 and s respectively.

$\therefore \vec{OP} = \vec{\gamma}(0)$

$\vec{OQ} = \vec{\gamma}(s)$

$\therefore \vec{PQ} = \vec{OQ} - \vec{OP}$

$= \vec{\gamma}(s) - \vec{\gamma}(0)$

Let $\vec{t} = \vec{\gamma}'(0)$ be the unit tangent vector

Let $\vec{R}$ be the position vector of the current point P on the plane then $PR = R - \vec{\gamma}(0)$ & $\vec{\gamma}'(0)$, $PQ = \vec{\gamma}(s) - \vec{\gamma}(0)$ are lie in the same plane

The eqn of the plane is

$[R - \vec{r}(0), \vec{r}'(0), \vec{r}(s) - \vec{r}(0)] = 0 \longrightarrow (1)$

also by taylor's thm, we have

$\vec{r}(s) = \vec{r}(0) + \frac{s}{1!}\vec{r}'(0) + \frac{s^2}{2!}\vec{r}''(0) + O(s^3)$

$\vec{r}(s) - \vec{r}(0) = s\vec{r}'(0) + \frac{s^2}{2}\vec{r}''(0) \longrightarrow (2)$

sub (2) in (1), we get

$(1) \Rightarrow [\vec{R} - \vec{r}(0), \vec{r}'(0), s\vec{r}'(0) + \frac{s^2}{2}\vec{r}''(0)] = 0$

$\Rightarrow [\vec{R} - \vec{r}(0), \vec{r}'(0), s\vec{r}'(0)] + [\vec{R} - \vec{r}(0), \vec{r}'(0), \frac{s^2}{2}\vec{r}''(0)] = 0$

$\Rightarrow s\underbrace{[\vec{R} - \vec{r}(0), \vec{r}'(0), \vec{r}'(0)]}_{0} + \frac{s^2}{2}[\vec{R} - \vec{r}(0), \vec{r}'(0), \vec{r}''(0)] = 0$

$\Rightarrow \frac{s^2}{2}[\vec{R} - \vec{r}(0), \vec{r}'(0), \vec{r}''(0)] = 0$

$[\vec{R} - \vec{r}(0), \vec{r}', \vec{r}''] = 0$

$[\vec{R} - \vec{r}, \vec{r}', \vec{r}''] = 0$ which is the required eqn.

* Theorem:

show that if a curve is given in terms of the General Parameter u then the equation of the oscillating plane is

$$[\vec{R} - \vec{r}(u), \dot{\vec{r}}(u), \ddot{\vec{r}}(u)] = 0.$$

Proof:

$k.t.T$

The eqn of the oscillating plane is $[R - \vec{r}, \vec{r}', \vec{r}''] = 0$

L.k.T

$$\vec{r}' = \frac{dr}{ds} = \frac{dr/du}{ds/du} = \frac{\dot{r}}{\dot{s}}$$

$$\vec{r}'' = \frac{d}{ds}(\vec{r}')$$

$$= \frac{d}{ds}\left(\frac{\dot{r}}{\dot{s}}\right)$$

$$= \frac{d}{du}\left(\frac{\dot{r}}{\dot{s}}\right)\frac{du}{ds}$$

$$= \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^2} \cdot 1/\dot{s}$$

$$= \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^3} \longrightarrow (2)$$

sub ② & ③ in ①,

we get

$$\left[R - \vec{r}, \frac{\dot{r}}{\dot{s}}, \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^3}\right] = 0$$

$$\Rightarrow \left[R - \vec{r}, \frac{\dot{r}}{\dot{s}}, \frac{\dot{s}\ddot{r}}{\dot{s}^3}\right] - \left[R - \vec{r}, \frac{\dot{r}}{\dot{s}}, \frac{\dot{r}\ddot{s}}{\dot{s}^3}\right] = 0.$$

$$\Rightarrow \frac{\dot{s}}{\dot{s}^4}\left[R - \vec{r}, \dot{r}, \ddot{r}\right] - \frac{\ddot{s}}{\dot{s}^4}\left[R - \vec{r}, \dot{r}, \dot{r}\right] = 0.$$

$$\Rightarrow \frac{\dot{s}}{\dot{s}^4}\left[R - \vec{r}, \dot{r}, \ddot{r}\right] = 0$$

$$\Rightarrow \left[R - \vec{r}, \dot{r}, \ddot{r}\right] = 0$$

7/7/14 Theorem:

Prove that cartesian equation of the osculating plane is.

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Proof:

W.K.T the equations of the osculating plane is $[R - \vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}] = 0 \longrightarrow ①$

Let $\vec{R} = x(u)\vec{i} + y(u)\vec{j} + z(u)\vec{k}$

$$\left.\begin{aligned} \vec{r} &= x(u)\vec{i} + y(u)\vec{j} + z(u)\vec{k} \\ \dot{\vec{r}} &= \dot{x}(u)\vec{i} + \dot{y}(u)\vec{j} + \dot{z}(u)\vec{k} \\ \ddot{\vec{r}} &= \ddot{x}(u)\vec{i} + \ddot{y}(u)\vec{j} + \ddot{z}(u)\vec{k} \end{aligned}\right] \rightarrow (2)$$

Note:

By using (2) eqns (1) can be written as

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Problem:

[Find the eqns of the osculating plane at a general point on the cubic curve given by $r = (u, u^2, u^3)$] and Show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these three points.

Soln:-

Given $\vec{r} = (u, u^2, u^3)$

W.K.T the eqn of the osculating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0 \longrightarrow (1)$$

Here $x = u,\ y = u^2,\ z = u^3$

$\dot{x} = 1,\ \dot{y} = 2u,\ \dot{z} = 3u^2$

$\ddot{x} = 0,\ \ddot{y} = 2,\ \ddot{z} = 6u$

① becomes

$$\begin{vmatrix} x-u & y-u^2 & z-u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 0$$

$\Rightarrow (x-u)[12u^2-6u^2] - (y-u^2)[6u-0] +$

$[z-u^3](2-0) = 0$

$\Rightarrow (x-u)(6u^2) - (y-u^2)6u + (z-u^3)2 = 0$

$\Rightarrow 6u^2x - 6u^3 - 6uy + 6u^3 + 2z - 2u^3 = 0$

$\Rightarrow 6u^2x - 6uy + 2z - 2u^3 = 0$

$\Rightarrow 2[3u^2x - 3uy + z - u^3] = 0$

$\Rightarrow 3u^2x - 3uy + z - u^3 = 0$

which is the required eqn of the osculating plane.

(ii)

Let u_1, u_2, u_3 be any three points on the given curve then the osculating plane at these three points are

$3u_1^2x - 3u_1y + z - u_1^3 = 0 \longrightarrow$ ②

$3u_2^2x - 3u_2y + z - u_2^3 = 0 \longrightarrow$ ③

$3u_3^3x - 3u_3y + z - u_3^3 = 0 \longrightarrow$ ④

Suppose that the planes ②, ③ & ④ at the points (x_0, y_0, z_0) we have

$3u_2^2x_0 - 3uy_0 + z_0 - u^3 = 0$

$\Rightarrow u^3 - 3u^2x_0 + 3uy_0 - z_0 = 0 \longrightarrow$ ⑤

Let the eqn of the plane passes through the three points (ur, ur^2, ur^3)

$\lambda = (1, 2, 3)$ be $Ax + By + Cz = 1$ $\longrightarrow$ (6)

$\therefore$ (u_1, u_2, u_3) are the roots of the eqn.

$Au + Bu^2 + Cu^3 - 1 = 0$ $\longrightarrow$ (7)

Comparing (6) & (7) we have,

$$\frac{A}{3y_0} = \frac{B}{-3x_0} = \frac{C}{1} = \frac{1}{z_0}$$

$$\frac{A}{3y_0} = \frac{1}{z_0} \Rightarrow A = \frac{3y_0}{z_0}$$

$$\frac{B}{-3x_0} = \frac{1}{z_0} \Rightarrow B = \frac{-3x_0}{z_0}$$

$$\frac{C}{1} = \frac{1}{z_0} \Rightarrow C = \frac{1}{z_0}$$

(6) becomes

$$\frac{3y_0}{z_0} x - \frac{3x_0}{z_0} y + \frac{1}{z_0} z = 1$$

$$\Rightarrow 3y_0 x - 3x_0 y + z = z_0$$

$$(3y_0)x - (3x_0)y + z - z_0 = 0$$

Which is the required eqn of the planes determined by the points (ur, ur^2, ur^3)

Problem:

Find the eqn of the osculating plane at the point on the helix.

$x = a\cos u$, $y = a\sin u$, $z = bu$.

Soln:-

Given $\vec{r} = (a\cos u, a\sin u, bu)$

W.K.T

The eqns of the osculating plane is

$$\begin{vmatrix} x-x & y-y & z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0 \longrightarrow (1)$$

Here $x = a\cos u$, $y = a\sin u$, $z = bu$

$\dot{x} = -a\sin u$, $\dot{y} = a\cos u$, $\dot{z} = b$

$\ddot{x} = -a\cos u$, $\ddot{y} = -a\sin u$, $\ddot{z} = 0$

(1) becomes

$$\begin{vmatrix} x-a\cos u & y-a\sin u & z-bu \\ -a\sin u & a\cos u & b \\ -a\cos u & -a\sin u & 0 \end{vmatrix} = 0$$

$\Rightarrow (x-a\cos u)\,[0+ab\sin u] - (y-a\sin u)\,[0+ab\cos u] + (z-bu)\,[a^2\sin^2 u + a^2\cos^2 u] = 0$

$\Rightarrow (x-a\cos u)(ab\sin u) - (y-a\sin u)(ab\cos u) + (z-bu)a^2 = 0$

$\Rightarrow abx\sin u - a^2b\sin u\cos u - aby\cos u + a^2b\sin u\cos u + za^2 - a^2bu = 0$

$\Rightarrow a\,(bx\sin u - by\cos u + za - abu) = 0$

$\Rightarrow bx\sin u - by\cos u + za - abu = 0$

which is the required equation.

8/7/14. Problem:-

point of inflexion $\vec{r}'' = 0$ (second derivative equal to 0)

Show that when the curve is analytic, obtain a definite osculating plane at a point of inflexion P unless the curve is a st-line.

Proof:

We have. $\vec{t} = \vec{r}'$ and $t \cdot t = 1$

$\vec{t} \cdot \vec{t} = 1$

$\therefore \vec{r}' \cdot \vec{r}' = 1 \longrightarrow (1)$

Diff (1) w.r.to 's'

$\vec{r}' \cdot \vec{r}'' + \vec{r}'' \cdot \vec{r}' = 0$

$\Rightarrow 2\,\vec{r}' \cdot \vec{r}'' = 0$

$\Rightarrow \vec{r}' \cdot \vec{r}'' = 0 \longrightarrow (2)$

Diff (2) w.r.to 's' we get

$\vec{r}' \cdot \vec{r}''' + \vec{r}'' \cdot \vec{r}'' = 0$

$\Rightarrow \vec{r}' \cdot \vec{r}''' = 0 \longrightarrow (3)$

Since $\vec{r}'' = 0$ due to point of inflexion.

[At which point P, $\vec{r}'' = 0$. that point is called point of inflexion.]

Case (i):

If $\vec{r}''' \neq 0$

then $\vec{r}$ and $\vec{r}'''$ are lenearly independent

$\therefore$ The osculating plane becomes

$[R - \vec{r}, \vec{r}', \vec{r}'''] = 0$.

case (ii):

If $\vec{r}''' = 0$.

Again diff ③ w.r.to 's' we get

$\vec{r}' \vec{r}^{IV} + \vec{r}''' . \vec{r}'' = 0$.

ie) $\vec{r}' \vec{r}^{IV} = 0 \quad [\because \vec{r}''' = 0]$

In General $\vec{r}' . \vec{r}^{k} = 0$

If $\vec{r}^{k} \neq 0$ for $k \geq 2$ then the eqn of osculating plane is

$[R - \vec{r}, \vec{r}', \vec{r}^{k}] = 0$.

If $\vec{r}^{k} = 0$ for $k \geq 2$, then $\vec{r}'' = 0$.

ie) $\vec{r}' = a$ constant

ie) $t = c$

The curve is a st. line. Since the curve is analytic.

Hence at a point of inflexion. even a curve of class ∞ need not Passes on osculating plane.

Definition:-

(i) Normal Plane :-

orthogonal (or) perpendicular

The plane through P which is normal to the tangent at P is called the normal plane at a point P on the curve.

The eqn is

$(R - \vec{r})\vec{r}' = 0$ (or)

$$\boxed{(R - \vec{r}) \cdot \vec{t} = 0}$$

Definition:-

(ii) Principal Normal

The Line of intersection of the normal plane and the osculating plane at P is the Principal normal at P.

Definition:

(iii) Binormal

The normal which is ⊥ to the osculating plane at a point is called the binormal.

Note:

The unit vector along the Principal normal and binormal are denoted by $\bar{n}$ and $\bar{b}$.

The unit vector are $\vec{t}, \vec{n}, \vec{b}$. Also

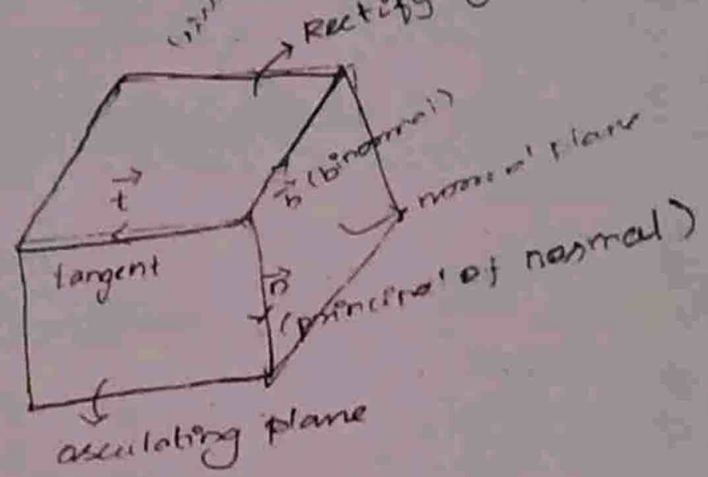
$\vec{t} = \vec{n} \times \vec{b}$, $\vec{b} = \vec{t} \times \vec{n}$, $\vec{n} = \vec{b} \times \vec{t}$

$\vec{t} \cdot \vec{n} = 0$, $\vec{n} \cdot \vec{b} = 0$; $\vec{b} \cdot \vec{t} = 0$.

i) The osculating plane containing $\vec{t}$ and $\vec{n}$ and its equal is $(R - \vec{r}) \cdot \vec{b} = 0$.

ii) The normal plane contains $\vec{n}$ and $\vec{b}$ and its equal is $(R - \vec{r}) \cdot \vec{t} = 0$.

iii) The ~~principal plane~~ Rectifying plane contains $\vec{n}$ and $\vec{t}$ and its equal is $(R - \vec{r}) \vec{n} = 0$.



Definition:

Curvature

The arc rate at which the tangent changes direction as P moves along the curve is the curvature of the curve and is denoted by k.

By definition $|k| = |t'|$.

k

Definition:-

Torsion.

As P moves along a curve the ~~ax~~ rate at which the osculating plane turns about the tangent is called the torsion of the curve and is denoted by τ

Note:

τ is determined by both in magnitude and direction.

SERRET- FRENET FORMULA:

Prove that

i) $\frac{d\vec{t}}{ds} = k\vec{n}$

ii) $\frac{d\vec{n}}{ds} = \tau\vec{b} - k\vec{t}$

(iii) $\frac{d\vec{b}}{ds} = -\tau\vec{n}$

Proof:

i) To prove: $\frac{dt}{ds} = k\vec{n}$

W.K.T

$$\vec{r}' = \frac{dr}{ds} = \frac{dr}{du}\cdot\frac{du}{ds} = \dot{\vec{r}}\vec{u}'$$

$$\vec{r}'' = \frac{d}{ds}(\vec{r}') = \frac{d}{ds}(\dot{\vec{r}}\vec{u}')$$

$$\vec{r}'' = \dot{\vec{r}}\frac{du'}{ds} + \frac{d\dot{\vec{r}}}{ds}\cdot u' \quad \Rightarrow \frac{d\dot{r}}{du}\frac{du}{ds}$$

$$= \dot{\vec{r}}u'' + \ddot{\vec{r}}u'.u'$$

$$= \dot{\vec{r}}u'' + \ddot{\vec{r}}u'^2$$

$\therefore \vec{r}''$ lie on the osculating plane $\quad t \cdot t = 1$

W.K.T

t is a unit tangent vector and $t^2 = 1$.

We know that $\vec{t}\cdot\vec{t} = 1$ —— ①

Diff ① w.r.to s

$$\vec{t}\frac{dt}{ds} + \vec{t}\frac{dt}{ds} = 0$$

$$\Rightarrow 2\vec{t}\frac{dt}{ds} = 0$$

$$\Rightarrow 2\vec{t}\,\vec{t}' = 0$$

$$\Rightarrow 2\vec{t}\,\vec{r}'' = 0$$

$$\Rightarrow \vec{t}\,\vec{r}'' = 0$$

i.e) $\vec{r}''$ is $\perp^{\circ}$ to $\vec{t}$ —— ②

from ① & ②

$\vec{r}''$ coincides with $\vec{n}$

i.e) $\vec{r}'' = k\vec{n}$. $[\because |\vec{r}''| = k]$

$$\Rightarrow \frac{d\vec{t}}{ds} = k\vec{n}.$$

Hence (i) is proved.

$\vec{t}\,' = k\vec{n}$

$\vec{t}\,\vec{t}\,' + \vec{t}\,\vec{t}\,' = 0$

$2\vec{t}\,\vec{t}\,' = 0$

$\vec{t}\,\vec{t}\,' = 0$

$\vec{t}\,' \propto \vec{n}$

$\vec{t}\,' = |k|\vec{n}$

ie) We know that $\vec{n} = \vec{b} \times \vec{t}$

Differentiate with respect to 's'.

$$\frac{d\vec{n}}{ds} = \frac{d\vec{b}}{ds} \times \vec{t} + \vec{b} \frac{dt}{ds}$$

$$= (-\tau n \times \vec{t}) + (\vec{b} \times k\vec{n})$$

$$= -\tau(n \times \vec{t}) + k(\vec{b} \times \vec{n})$$

$$= -\tau(-\vec{b}) + k(-\vec{t})$$

$$= \tau\vec{b} - k\vec{t}$$

Hence (ii) is proved.

iii) To prove : $\frac{d\vec{b}}{ds} = -\tau\vec{b}$

W.K.T $\vec{t}$ is $\perp^{r}$ to $\vec{b}$

ie) $\vec{t} \cdot \vec{b} = 0$.

Diff with respect to s,

We have,

$$\Rightarrow \vec{t} \cdot \frac{d\vec{b}}{ds} + \vec{b} \cdot \frac{d\vec{t}}{ds} = 0$$

$$\Rightarrow \vec{t} \frac{d\vec{b}}{ds} + \vec{b}\, \vec{r}'' = 0$$

$$\Rightarrow \vec{t} \cdot \frac{d\vec{b}}{ds} + \vec{b} \cdot k\vec{n} = 0$$

$$\Rightarrow \vec{t} \frac{d\vec{b}}{ds} + k(\vec{b} \cdot \vec{n}) = 0$$

$$\therefore t \frac{db}{ds} + c = 0$$

$$\Rightarrow \vec{t} \frac{d\vec{b}}{ds} = 0$$

$\therefore \frac{d\vec{b}}{ds}$ is $\perp^r$ to $\vec{t} \longrightarrow (3)$

W.K.T $\vec{b}$ is unit binormal.

$\vec{b} \cdot \vec{b} = 1.$

Diff w.r.to 's'

$\vec{b} \cdot \frac{d\vec{b}}{ds} + \frac{d\vec{b}}{ds} \cdot \vec{b} = 0$

$\Rightarrow \vec{b} \cdot \vec{b}' + \vec{b}' \vec{b} = 0$

$\Rightarrow 2\vec{b}\vec{b}' = 0$

$\Rightarrow \vec{b}\vec{b}' = 0$

$\Rightarrow \vec{b}'$ is $\perp^r$ to $\vec{B} \longrightarrow (4).$

from (3) & (4) we have

$\frac{d\vec{b}}{ds}$ coincide with $\vec{n}$

ie) $\left|\frac{d\vec{b}}{ds}\right| = \tau \cdot \vec{n}$

$\therefore \frac{db}{ds} = -\tau \cdot \vec{n}$

Hence (iii) is proved.

Problem:

Show that the necessary and sufficient condition that a curve to be st. line is that $k = 0$ at all points

Proof: Necessary Part.

The equation of a straight line

$$\vec{r} = a\vec{s} + \vec{b} \rightarrow (1).$$

Where $\vec{a}$ and $\vec{b}$ are constant vectors.

Diff. (1) with respect to "s"

We have,

$(\because \vec{r}' = \vec{t})$

$$\frac{dr}{ds} = a \Rightarrow \vec{r}' = a$$

ie) $\vec{t} = a \rightarrow (2)$

Diff. (2) with respect to "s"

$$\frac{d\vec{t}}{ds} = 0 \Rightarrow \vec{t}' = 0$$

But $\vec{t}' = kn \rightarrow (3)$

[By S.F formula]

$\therefore kn = 0$

$\Rightarrow k^2 = 0$

$k = 0$ at all points

Sufficient Part.

If $k = 0$ then (3) becomes,

$$\vec{t}' = 0$$

ie) $\vec{r}'' = 0 \rightarrow (4)$

Inting (4) we get

$\vec{r}' = a$ (const) $\rightarrow (5)$

Integrating (5) w.r.to "s"

$r = as + b \rightarrow (6)$

Where a & b are constants

Eqn (6) represents a st. line

Theorem:-

Prove that a necessary & sufficient condition that a curve to be a plane curve is that $\tau = 0$ at all points.

Proof:- Necessary Part.

Assume that a curve is a plane curve

To prove:

$\tau = 0$ at all points

Since the curve is a plane curve, the tangent and normal at all points are also lie on the plane.

$\therefore$ The plane is an osculating plane for all points.

$\Rightarrow$ The binormal vector is same at all points.

The binormal vector is constant.

$\Rightarrow \vec{b} = \vec{c}$ where $\vec{c}$ is a constant Vector.

$\Rightarrow \frac{d\vec{b}}{ds} = 0$

$\Rightarrow \left|\frac{db}{ds}\right| = 0$

$\Rightarrow \tau = 0$ at all points.

Sufficient Part:-

Assume that $\tau = 0$ at all Points.

To prove:-

The curve γ is a plane curve.

Now $\tau = 0$

$\Rightarrow \left|\frac{db}{ds}\right| = 0$

$\Rightarrow |b'| = 0$

$\Rightarrow b' = 0$

Igrating w.r.to s, we get

$\vec{b} = \vec{c}$, where c is a constant Vector.

$\Rightarrow$ The binormal vector is same at all points.

ant

$\Rightarrow$ The plane is an osculating plane.

$\Rightarrow$ The tangent plane & normal at all points are also lie on the plane.

$\therefore$ The curve is a plane curve.

17/14 Theorem:

1) Show that the necessary & sufficient condition for the curve to be a plane curve is $[r', r'', r'''] = 0$.

NECESSARY CONDITION

Proof: Assume that the given curve is plane curve.

W.K.T

$$r' = \frac{dr}{ds}$$

$$\Rightarrow r' = t \longrightarrow (1)$$

Diff (1) w.r.to s.

$$\Rightarrow r'' = t'$$

But W.K.T $\frac{dt}{ds} = t' = kn \longrightarrow (2)$ [S-F formula]

Diff (2) w.r.to s.

$$r''' = \frac{d}{ds}(r'')$$

$$= \frac{d}{ds}(kn)$$

$\Rightarrow \frac{d}{ds} r''' = \frac{d}{ds}(kn)$

$\Rightarrow r''' = \frac{d}{ds} k \cdot n + k \cdot \frac{dn}{ds}$

$\Rightarrow r''' = k' \bar{n} + k(\tau \bar{b} - k t)$

$\Rightarrow r''' = k' \cdot \bar{n} + k\tau \bar{b} - k^2 \bar{t} \longrightarrow (3)$

Now,

$$\begin{array}{cc} & \begin{array}{ccc} t & n & b \end{array} \\ [r', r'', r'''] = & \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ -k^2 & k' & k\tau \end{vmatrix} \end{array}$$

$= 1(k^2\tau - 0) - 0(0(k\tau) - 0) + 0[0(k') + k^2]$

$= k^2\tau.$

Given $[r', r'', r'''] = 0 \rightarrow (4)$

$\therefore (4) \Rightarrow k^2\tau = 0$

Either $k = 0$ (or) $\tau = 0$

Suppose $\tau \neq 0$ at some

Points of the curve then

in the neighbourhood of these

point $\tau \neq 0$.

Therefore,

We have $\cancel{k}=0$. and Assume $[\vec{r}', \vec{r}'', \vec{r}''']=0$ sufficient part:

Given curve is a st. line. i.e. plane curve $k\tau=0 \Rightarrow k=0$ or $\tau=0$

Which is $\Rightarrow \Leftarrow$ to our assumption.

$\therefore \tau=0$ at all points and

Given curve is a plane curve. st. line. If $k=0$ the curve is st line.

If $\tau=0$ the curve plane [thm]

SUFFICIENT CONDITION

Assume that $\tau=0$ at all

Points.

$\Rightarrow$ The given curve is a plane curve

$\therefore$ We have $[r', r'', r''']=0$.

Which is the sufficient condition for the plane curve.

Theorem:

Show that $[\dot{r}, \ddot{r}, \dddot{r}]=0$ is the necessary & sufficient condition for a curve to be a plane curve

Proof: Necessary part

Assume that given curve is a plane curve.

W.K.T

$$r' = \frac{dr}{ds}$$

$\frac{dr}{ds}$; $\frac{du}{ds}$; $\frac{dr}{du}$; $\dot{r}/s$

$$r' = \frac{dr}{du} \cdot \frac{du}{ds}$$

$$r' = \dot{r} \cdot u' \longrightarrow (1)$$

$$r'' = \frac{d}{ds}(\dot{r} \cdot u')$$

$$= \frac{d\dot{r}}{ds} \cdot u' + \dot{r} \cdot \frac{d}{ds}(u')$$

$$= \frac{d\dot{r}}{du} \cdot \frac{du}{ds} u' + \dot{r} u''$$

$$r'' = \ddot{r} u'^2 + \dot{r} u'' \longrightarrow (2)$$

$$r''' = \frac{d}{ds}\left[\ddot{r} u'^2 + \dot{r} u''\right]$$

$$= \frac{d}{ds}\left[\ddot{r} u'^2\right] + \frac{d}{ds}\left[\dot{r} u''\right]$$

$$= \frac{d\ddot{r}}{ds} u'^2 + \ddot{r} \frac{du'^2}{ds} + \frac{d\dot{r}}{ds} \cdot u'' + \dot{r} \frac{du''}{ds}$$

$$= \frac{d\ddot{r}}{du} \frac{du}{ds} \cdot u'^2 + \ddot{r}\, 2u'u'' +$$

$$\frac{d\dot{r}}{du} \frac{du}{ds} u'' + \dot{r} u'''$$

$r''' = \dddot{r} \cdot u'^3 + \ddot{r}\, 2u'u'' + \ddot{r} u'u'' + \dot{r} u'''$

$r''' = \dddot{r} u'^3 + 3\ddot{r} u'u'' + \dot{r} u''' \longrightarrow (3)$

W.K.T the given curve is a plane curve only when $[r', r'', r'''] = 0$.

$\therefore [r', r'', r'''] = [\dot{r}u', \ddot{r}u'^2 + \dot{r}u'', \dddot{r}u'^3 + 3\ddot{r}u'u'' + \dot{r}u''']$

$= [\dot{r}u', \ddot{r}u'^2 + \dot{r}u'', \dddot{r}u'^3] + [\dot{r}u', \ddot{r}u'^2 + \dot{r}u'', 3\ddot{r}u'u''] + [\dot{r}u', \ddot{r}u'^2 + \dot{r}u'' + \dot{r}u''']$

$= [\dot{r}u', \ddot{r}u'^2, \dddot{r}u'^3] + [\dot{r}u', \dot{r}u'', \dddot{r}u'^3] + [\dot{r}u', \ddot{r}u'^2, 3\ddot{r}u'u''] + [\dot{r}u', \dot{r}u'', 3\ddot{r}u'u''] + [\dot{r}u', \ddot{r}u'^2, \dot{r}u'''] + [\dot{r}u', \dot{r}u', \dot{r}u''']$

$= [\dot{r}u', \ddot{r}u'^2, \dddot{r}u'^3] + 0 + 0 + 0 + 0 + 0$.

$= [\dot{r}u', \ddot{r}u'^2, \dddot{r}u'^3]$

$= u'^6 [\dot{r}, \ddot{r}, \dddot{r}]$

$\therefore [\dot{r}, \ddot{r}, \dddot{r}] = \frac{1}{u'^6} [r', r'', r''']$

$= (u')^{-6} [r', r'', r''']$

$\Rightarrow [\dot{r}, \ddot{r}, \dddot{r}] = (u')^{-6} k^2 \tau$.

$[\because (r', r'', r''' = k^2 \tau)]$

Given that

$[\dot{r}, \ddot{r}, \dddot{r}] = 0 \rightarrow (4)$

$\therefore$ eqn (4) becomes.

$(u')^{-6} k^2 \tau = 0$.

$\Rightarrow k^2 \tau = 0$.

either $\tau = 0$ or $k = 0$

suppose $\tau \neq 0$ at all points

of the curve then $k = 0$

ie) the given curve is st. line

which is $\Rightarrow \epsilon$ to our assumption

$\therefore \tau = 0$ at all points.

[The binormal is a constant

$\Rightarrow b = c \Rightarrow \frac{db}{ds} = 0$.

$\Rightarrow \left|\frac{db}{ds}\right| = 0$

$\Rightarrow |b'| = 0$

$\Rightarrow |\tau| = 0$

$\tau = 0$]

Sufficient condition:-

Assume that $\tau = 0$.

$\Rightarrow$ The given curve is a plane curve.

$\therefore [\dot{r}, \ddot{r}, \dddot{r}] = 0$.

which is the sufficient codition for the curve to the plane.

14/7/14. Theorem:

Prove that $|K| = |\vec{r}' \times \vec{r}''| = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$.

Proof:

We know that $\vec{r}' = t$

$r'' = t' = k\vec{n}$.

$r' \times r'' = \vec{t} \times k\vec{n}$

$= k(\vec{t} \times \vec{n})$.

$= k\vec{b}$.

$|r' \times r''| = |k\vec{b}| = |k|$ $[\because |b| = 1.]$.

Now, $r' = \dfrac{dr}{ds} = \dfrac{dr/du}{ds/du} = \dot{r}/\dot{s}$.

$r'' = d/ds(\dot{r}/\dot{s}) = d/ds(\dot{r}/\dot{s}) \cdot \dfrac{du}{ds}$

$$r'' = \frac{s'\ddot{r} - \dot{r}\ddot{s}}{s^{\cdot 3}}$$

$$\vec{r}' \times \vec{r}'' = \frac{\dot{r}}{\dot{s}} \times \left[\frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^3}\right]$$

$$= \left(\frac{\dot{r}}{\dot{s}}\right) \times \left(\frac{\dot{s}\ddot{r}}{\dot{s}^3}\right) - \left(\frac{\dot{r}}{\dot{s}}\right) \times \left(\frac{\dot{r}\ddot{s}}{\dot{s}^3}\right)$$

$$= \frac{\dot{s}}{\dot{s}^4}(\dot{r} \times \ddot{r}) - \frac{\ddot{s}}{\dot{s}^4}(\dot{r} \times \dot{r})$$

$$= \frac{\dot{r} \times \ddot{r}}{\dot{s}^3} - \frac{\ddot{s}(\dot{r} \times \dot{r})}{\dot{s}^4}$$

$$= \frac{\dot{r} \times \ddot{r}}{\dot{s}^3} \quad [\because \dot{r} \times \dot{r} = 0]$$

$$s = \int |\dot{\vec{r}}|\, du$$

$$\dot{s} = |\dot{\vec{r}}|$$

$$\therefore |\vec{r}' \times \vec{r}''| = \left|\frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3}\right|$$

$$= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{s}|^3}$$

$$= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \quad [\because \dot{s} = \dot{r}]$$

Hence. $|k| = |r' \times r''| = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$

Problem:

Prove that $\tau = \dfrac{[\vec{r}', \vec{r}'', \vec{r}''']}{|\vec{r}' \times \vec{r}''|}$

$$\tau = \frac{[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

Determine the Function $f(u)$. so that the curve given by $\vec{r} = [a\cos u, a\sin u, f(u)]$ should be a plane.

Soln:

Given that, $\vec{r} = (a\cos u, a\sin u, f(u))$ and also the given curve is a plane curve.

ie) $\tau = 0$.

We know that

$$\frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|} = \tau.$$

We have $\tau = 0$

$$\frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|} = 0.$$

$$\Rightarrow [\dot{r}, \ddot{r}, \dddot{r}] = 0.$$

$\vec{r} = (a\cos u, a\sin u, f(u))$

$\dot{r} = (-a\sin u, a\cos u, f'(u))$

$\ddot{r} = (-a\cos u, -a\sin u, f''(u))$

$\dddot{r} = (a\sin u, -a\cos u, f'''(u))$

$$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} -a\sin u & a\cos u & f'(u) \\ -a\cos u & -a\sin u & f''(u) \\ a\sin u & -a\cos u & f'''(u) \end{vmatrix}$$

$R_1 \to R_1 + R_3$

$$\Rightarrow [\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 0 & 0 & f(u)+f''(u) \\ -a\cos u & -a\sin u & f''(u) \\ a\sin u & -a\cos u & f'''(u) \end{vmatrix}$$

$$= f(u)+f'''(u)\,[a^2\cos^2 u + a^2\sin^2 u].$$

$$= a^2\,[f'(u) + f'''(u)] = 0.$$

$$\Rightarrow f'(u) + f'''(u) = 0.$$

Sing. We get

$$\Rightarrow f(u) + f''(u) = C$$

$$\Rightarrow f(u) + \frac{d^2}{du^2}[f(u)] = C$$

$$\Rightarrow f(u)\left[1 + \frac{d^2}{du^2}\right] = C.$$ This is the 2nd order D.E.

$$\Rightarrow (1+D^2)f(u) = C,$$

Auxillary eqn is

$$m^2+1 = 0.$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \pm i$$

$$\alpha = 0,\ \beta = 1$$

$$Cf = e^{0u}[A\cos u + B\sin u]$$

$$P.I = \frac{1}{f(D)}\,x = \frac{1}{D^2+1}\,c$$

$$P.I = \frac{1}{D^2+1} \, c \cdot e^{0u}$$

$= e \, [\, e^{a x}$. D replace by a].

$\therefore \; f(u) = C.I + P.I$

$= e^{0u}[A\cos u + B \sin u] + C.$

Q. Calculate the curvature & Torsion
(ii) the cubic curve is given by
(u, u^2, u^3)

We know that,

$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{r}|^3} \quad \& \quad \tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

Given,

$r = (u, u^2, u^3)$

$\dot{r} = (1, 2u, 3u^2)$.

$\ddot{r} = (0, 2, 6u)$

$\dddot{r} = (0, 0, 6)$.

Also, $[\dot{r}\ \ddot{r}\ \dddot{r}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix}$

$= 1(12-0) - 2u(0-0) + 3u^2(0-0)$

$= 12 - 0 + 0$

$= 12$.

$$\dot{r} \times \ddot{r} = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$

$$= \vec{t}\,(12u^2 - 6u^2) - \vec{n}\,(6u - 0) + \vec{b}\,(2 - 0)$$

$$= \vec{t}\,(6u^2) - \vec{n}\,(6u) + \vec{b}\,(2)$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{36u^4 + 36u^2 + 4}$$

$$= \sqrt{4(9u^4 + 9u^2 + 1)}$$

$$= 2\sqrt{9u^4 + 9u^2 + 1}$$

$$|\dot{\vec{r}}| = \sqrt{1 + 4u^2 + 9u^4}$$

$$= (1 + 4u^2 + 9u^4)^{1/2}$$

$$|\dot{\vec{r}}|^3 = (1 + 4u^2 + 9u^4)^{3/2}$$

$$k = \frac{2\sqrt{9u^4 + 9u^2 + 1}}{(1 + 4u^2 + 9u^4)^{3/2}}$$

Also,

$$[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 12.$$

$$|\vec{r}' \times \vec{r}''|^2 = \left[\sqrt{4(9u^4 + 9u^2 + 1)}\right]^2$$

$$= 4(9u^4 + 9u^2 + 1).$$

$$= 36u^4 + 36u^2 + 4.$$

$$\boxed{\tau = \frac{12}{36u^4 + 36u^2 + 4.}} \quad \frac{3}{9u^4 + 9u^2 + 1}. \;//$$

Find the co-ordinates of a point in terms of s.

Soln:

Let P be any point on the given curve.

Take O as the origin and be along axes Ox, Oy, Oz respectively $\vec{t}$, $\vec{n}$, $\vec{b}$ respectively.

Let x, y, z be the co-ordinates of the neighbouring points Q with Position vector $\vec{r} = x\cdot\vec{t} + y\cdot\vec{n} + z\vec{b}$ $\rightarrow (1)$

If the ~~class~~ curve is of class ≥ 4 then s denote the small arc length PQ.

By Taylor's thm, we have

$$r(s) = r(0) + \frac{s}{1!}\vec{r}'(0) + \frac{s^2}{2!}r''(0) + \frac{s^3}{3!}r'''(0) + \frac{s^4}{4!}r^{iv}(0) + O(s^5)$$

as $s \to 0$

$\longrightarrow (2)$

We have $\vec{r}(0) = 0$

$\vec{r}'(0) = t$

$\vec{r}''(0) = kn$

$\vec{r}'''(0) = k'\vec{n} + k\tau\vec{b} - k^2\vec{t}$

$$\vec{r}^{iv}(0) = \frac{d}{ds}(k'\vec{n} + k\tau\vec{b} - k^2\vec{t})$$

$$\vec{r}^{iv}(0) = k'\frac{dn}{ds} + \vec{n}\cdot k'' - \left(k^2\frac{dt}{ds} + \vec{t}\,2kk'\right)$$

$$+ \left(k\tau\frac{d\vec{b}}{ds} + k\vec{b}\tau' + \tau\vec{b}k'\right)$$

$$\vec{\gamma}^{iv}(0) = \kappa'(\tau\vec{b} - \kappa\vec{t}) + \vec{n}\kappa'' - \kappa^2(\kappa\vec{n})$$
$$- \vec{t}\,2\kappa\kappa' + \kappa\tau(-\tau\vec{b}) +$$
$$\kappa\vec{b}\tau' + \tau\vec{b}\kappa'$$

$$= \kappa'\tau\vec{b} - \kappa\kappa'\vec{t} + \vec{n}\kappa'' - \kappa^3\vec{n} -$$
$$\vec{t}\,2\kappa\kappa' - \kappa\tau^2\vec{b} +$$
$$\kappa\vec{b}\tau' + \tau\vec{b}\kappa'.$$

$$= 2\kappa'\tau\vec{b} - 3\kappa\kappa'\vec{t} + \vec{n}\kappa'' - \kappa^3\vec{n}.$$
$$- \kappa\tau^2\vec{b} + \kappa\vec{b}\tau'.$$

$$= \vec{b}(2\kappa'\tau - \kappa\tau^2 + \kappa\tau')$$
$$+ \vec{n}(\kappa'' - \kappa^3).$$
$$+ \vec{t}(-3\kappa\kappa') +$$

$$= -3\kappa\kappa'\vec{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\vec{n} +$$
$$(\kappa'\tau + 2\kappa\tau')\vec{b}$$

$$\gamma(s) = 0 + s\vec{t} + \frac{s^2}{2}\kappa n + \frac{s^3}{6}(-\kappa^2 t + \kappa' n + \kappa\tau b)$$
$$+ \frac{s^4}{24}\left[-3\kappa\kappa' t + (\kappa'' - \kappa^3 - \kappa\tau^2)\vec{n}) + (\kappa'\tau + 2\kappa\tau')b\right.$$
$$\left. + O(s^5)\right]$$

$$\left(s - \frac{s^3\kappa^2}{6} - \frac{3\kappa\kappa' s^4}{24}\right)\vec{t} +$$
$$\left[\frac{s^2\kappa}{2} + s^3/6\,\kappa' + s^4/24(\kappa'' - \kappa^3 - \kappa\tau^2)\right]\vec{n} +$$
$$\left[\frac{s^3\kappa\tau}{6} + (\kappa'\tau + 2\kappa\tau')s^4/24\right]\vec{b}$$

Hence

$$x = s - \frac{s^3 k^2}{6} - \frac{k k' s^4}{8}$$

$$y = \frac{s^2 k}{2} + \frac{s^3 k'}{6} + \frac{s^4}{24}\left(k'' - k^3 - k\tau^2\right)$$

$$z = \frac{s^3 k\tau}{6} + \left(k^2\tau' + 2k'\tau\right)s^4/24.$$

Prove that

i) $\lim_{s \to 0} \frac{2y}{x^2} = k$

ii) $\lim_{s \to 0} \frac{3z}{xy} = \tau$

iii) $\sqrt{x^2+y^2+z^2} = s\left[1 - \frac{k^2 s^2}{24}\right]$

Soln:

We know that

$$x = s - \frac{s^3 k^2}{6} - \frac{k k' s^4}{8}$$

$$y = \frac{s^2 k}{2} + \frac{s^3 k'}{6} + \frac{s^4}{24}(k'' - k^3 - k\tau^2$$

$$z = \frac{s^3 k \tau}{6} + (k^2 \tau' + 2k'\tau)\, s^4/24$$

Now $\sqrt{x^2+y^2+z^2} = \Big\{\Big[s - \frac{s^3k^2}{3!} - \frac{3kk's^4}{4!} + O(s^5)\Big]^2 + \Big[\frac{s^2k}{2!} + \frac{s^3k'}{3!} + \frac{s^4}{4!}(-k\tau^2 - k^3 + k'') + O(s^5)\Big]^2 + \Big[\frac{s^3}{3!}(k\tau) + \frac{s^4}{4!}(k\tau' + 2k'\tau) + O(s^5)\Big]^2\Big\}^{1/2}$

$$= \Big\{\Big[s - \frac{s^3k^2}{3!}\Big]^2 + \Big[\frac{s^2k}{2!} + \frac{s^3k'}{3!}\Big]^2 + \Big[\frac{s^3}{3!}k\tau\Big]^2\Big\}^{1/2}$$ + omitting higher powers.

$$= \Big[s^2 + \frac{s^6k^4}{36} - \frac{2s^4k^2}{6} + \frac{s^4k^2}{4} + \frac{s^6k'^2}{36} + \frac{2s^5kk'}{12} + \frac{s^6}{36}k^2\tau^2\Big]^{1/2}$$

$$= \Big[s^2 + \frac{s^6k^4}{36} + \frac{s^6}{36}k^2\tau^2 - \frac{s^4k^2}{3} + \frac{s^4k^2}{4} + \frac{s^6k'^2}{36} + \frac{s^5kk'}{6}\Big]^{1/2}$$

$$= \Big[s^2 - \frac{s^4k^2}{3} + \frac{s^4k^2}{4}\Big]^{1/2},$$ omitting higher powers

$$= \Big[\frac{12s^2 - 4s^4k^2 + 3s^4k^2}{12}\Big]^{1/2}$$

$$= \Big[\frac{12s^2 - s^4k^2}{12}\Big]^{1/2}$$

$$= s\Big[1 - \frac{s^2k^2}{12}\Big]^{1/2}$$

$$= s\left[1 - \frac{1}{2}\left(\frac{s^2k^2}{12}\right) + \frac{1}{2}\left(\frac{s^2k^2}{12}\right)^2 - \dots\right]$$

$$= s\left[1 - \frac{1}{2}\left(\frac{s^2k^2}{12}\right) + \frac{1}{2}\left(\frac{s^4k^4}{144}\right)\right]$$

omitting higher powers.

$$= s\left[\cancel{1} - \frac{1}{2}\left(\frac{s^2k^2}{12}\right)\right]\left[\cancel{1} + \frac{s^2k^2}{12}\right]$$

$$= s\left[1 - \frac{s^2k^2}{24}\right].$$

$$\therefore \sqrt{x^2+y^2+z^2} = s\left[1 - \frac{s^2k^2}{24}\right].$$

(i) $x^2 = \left(s - \frac{s^3k^2}{6} - \frac{3kk's^4}{24}\right)^2$

$= \left(s - \frac{s^3k^2}{3}\right)$ omitting higher power.

$$x^2 = s^2 + \frac{s^6k^4}{36} - \frac{2s^4k^2}{6}.$$

$2y = 2\left(\frac{s^2k}{2} + \frac{s^3}{6}k' + \frac{s^4}{24}\right)(k'' - k^3 - k\tau^2)$

$= 2\frac{s^2k}{2} + 2\frac{s^3}{6}k'$ (omitting highap

$$2y = s^2k + s^3k'.$$

$$\frac{2y}{x^2} = \frac{s^2k + s^3k'}{s^2 + \frac{s^6k^4}{36} - \frac{2s^4k^2}{6}}$$

$= \frac{s^2k + s^3k'}{s^2}$ (omitting higher powers

$$\frac{2y}{x^2} = k + sk'.$$

$$\lim_{s\to 0} \frac{2y}{x^2} = \lim_{s\to 0} k + sk'.$$

ie) $\lim_{s\to 0} \frac{2y}{x^2} = k.$

(ii)

$$xy = \left(s - \frac{s^3k^2}{6} - \frac{3kk's^4}{24}\right) \times \left(\frac{s^2k}{2} + \frac{s^3}{6}k' + \frac{s^4}{24}(k'' - k^3 - k\tau^2)\right)$$

$$= \left(s - \frac{s^3k^2}{6}\right) \times \left(\frac{s^2k}{2} + \frac{s^3}{6}k'\right)$$

omitting higher powers.

$$= \frac{s^3k}{2} - \frac{s^5k^3}{12} + \frac{s^4k'}{6} - \frac{s^6k'k^2}{36}$$

$xy = \frac{s^3k}{2}$ [omitting higher powers]

$$3z = 3\left(\frac{s^3k\tau}{6} + (k\tau' + 2k'\tau)\frac{s^4}{24}\right)$$

$3z = \frac{3s^3k\tau}{6}$ [omitting higher powers]

$$3z = \frac{s^3k\tau}{2}$$

$$\frac{3z}{xy} = \frac{s^3k\tau/2}{s^3k/2} = \frac{s^3k\tau}{2} \times \frac{2}{s^3k} = \tau$$

$$\lim_{s \to 0} \frac{3z}{xy} = \lim_{s \to 0} \tau$$

$$\lim_{s \to 0} \frac{3z}{xy} = \tau$$

$V = m$

Show that the projection of the curve near P on the osculating plane is approximately the curve $z=0,\ y=\frac{1}{2}\kappa x^2$; its projection on the rectifying plane is approximately $y=0,\ z=\frac{1}{6}(\kappa\tau x^3)$ and which projection on the normal plane is approximately $x=0,\ z^2=\frac{2}{9\kappa}(\tau^2 y^3)$.

Soln:

We know that P is any point on the curve and $P(x,y)$ interms of s are

$$x = s - \frac{s^3\kappa^2}{6} - \frac{\kappa\kappa' s^4}{4!} + O(s^5)$$

$$y = \frac{s^2\kappa}{2!} + \frac{s^3\kappa'}{3!} + \frac{s^4}{4!}(-\kappa\tau^2 - \kappa^3 + \kappa'')$$

$$z = \frac{s^3}{3!}(\kappa\tau) + \frac{s^4}{4!}(\kappa\tau' + 2\kappa'\tau)$$

Let us consider ~~8~~ $x = s$,

$y = \frac{s^2 k}{2}$, $z = \frac{s^3}{31}(k\tau)$

Suppose we project the curve and the osculating plane $z = 0$.

$\therefore y = \frac{s^2 k}{2}$,

$\Rightarrow y = \frac{x^2 k}{2}$ $[\because x = s]$

Hence the curve is $z = 0$, and

$y = \frac{x^2 k}{2}$.

Suppose we project the curve on the normal plane.

ie) $x = 0$

$$z = \frac{s^3}{6}(k\tau)$$

$$z^2 = \frac{s^6}{36} k^2 \tau^2.$$

$$z^2 = \frac{s^6}{36} k^2 \tau^2 \cdot \frac{8k}{8k}$$

$$= \frac{\tau^2 8}{36 k}\left(\frac{s^6 k^3}{8}\right).$$

$$= \frac{\tau^2}{k} \times 2/9 \left(y^3\right)$$

$$z = 2/9 \frac{\tau^2 y^3}{k}$$

∴ The curve is $x = 0$,

$$z^2 = \frac{2}{9}\frac{\tau^2 y^3}{k}.$$

Suppose we project the curve on the rectifying plane $y = 0$.

$$\therefore z^2 = \frac{s^3 k \tau}{6}.$$

$$z = \frac{x^3 k \tau}{6} \quad [\because x = s]$$

Hence the $y = 0$, $z = \frac{x^3 k \tau}{6}$.

(7) Show that the length of the common perpendicular "d" of the tangents at two near points distance "s" apart this approximately given by $d = \frac{k \tau x^3}{12}$.

Sln:-

Let γ be any curve.

Let P, Q be two points on the curve with the parameters 0 and s respectively.

$\overline{OP} = r(0)$, $\overline{OQ} = r(s)$.

Let the unit tangent vector at point P and Q are $r'(0)$ & $r'(s)$ respectively

Then $r'(0) \times r'(s)$ is the common $\perp^r$ to both $r'(0)$ & $r'(s)$.

ie) The length of the common $\perp^r$ is $d = \dfrac{[\vec{r}(s) - \vec{r}(0), \vec{r}'(s), \vec{r}'(0)]}{|\vec{r}'(s) \times \vec{r}'(0)|}$ $\rightarrow$ (a)

If the curve is of class ≥ 3.

then by Taylor's series.

$$r(s) = r(0) + \frac{s}{1!} r'(0) + \frac{s^2}{2!} r''(0) + \frac{s^3}{3!} r'''(0) + O(s^4),\ s \to 0 \rightarrow (1)$$

We know that

$\vec{r}'(0) = t$

$\vec{r}''(0) = t' = k\vec{n}$

$\vec{r}'''(0) = t'' = -k^2\vec{t} + k'\vec{n} + k\tau\vec{b}$

Sub r', r'', r''' in eqn (1)

$$(1) \Rightarrow r(s) = r(0) + \frac{s}{1!} t + \frac{s^2}{2!}(k\vec{n}) + \frac{s^3}{3!}(-k^2\vec{t} + k'\vec{n} + k\tau\vec{b}) + O(s^4)\quad s \to 0$$

$$\Rightarrow r(s) - r(0) = st + \frac{s^2}{2}(k\vec{n}) + \frac{s^3}{6}(-k^2\vec{t} + k'\vec{n} + k\tau\vec{b}) + O(s^4)\ \text{as } s \to 0$$

$$\Rightarrow \gamma(s)-\gamma(0) = s\vec{t} + \frac{s^2\kappa\vec{n}}{2} - \frac{\kappa^2\vec{t}s^3}{6} + \frac{\kappa'\vec{n}s^3}{6} + \frac{s^3}{6}\kappa\tau\vec{b} + O(s^4)$$
as $s \to 0$

$$= s\vec{t} - \frac{\kappa^2\vec{t}s^3}{6} + \frac{s^2\kappa\vec{n}}{2} + \frac{\kappa'\vec{n}s^3}{6} + \frac{s^3}{6}\kappa\tau\vec{b}, \text{ as } s\to 0.$$

$$= \vec{t}\left(s - \frac{\kappa^2 s^3}{6}\right) + \vec{n}\left(\frac{s^2\kappa}{2} + \frac{\kappa' s^3}{6}\right) + \vec{b}\left(\frac{\kappa\tau s^3}{6}\right), \text{as } s\to 0.$$

$$\gamma'(s) = \vec{t}\left(1 - \frac{3s^2\kappa^2}{6}\right) + \vec{n}\left(\frac{\cancel{2}s\kappa}{\cancel{2}} + \frac{3s^2\kappa'}{\cancel{6}}\right) + \vec{b}\left(\frac{3s^2\kappa\tau}{6}\right), \text{as } s\to 0.$$

$$= \vec{t}\left(1 - \frac{s^2\kappa^2}{2}\right) + \vec{n}\left(s\kappa + \frac{s^2\kappa'}{2}\right) + \vec{b}\left(\frac{s^2\kappa\tau}{2}\right),$$
as $s \to 0$

$$[\vec{\gamma}(s) - \vec{\gamma}(0), \vec{\gamma}'(s), \vec{\gamma}'(0)]$$

$$= \begin{vmatrix} s - \frac{\kappa^2 s^3}{6} & \frac{s^2\kappa}{2} + \frac{\kappa' s^3}{6} & \frac{\kappa\tau s^3}{6} \\ 1 - \frac{s^2\kappa^2}{2} & s\kappa + \frac{s^2\kappa'}{2} & \frac{s^2\kappa\tau}{2} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{\kappa^2 s^3}{6}(0) - \left(\frac{s^2\kappa}{2} + \frac{\kappa' s^3}{6}\right)\left(-\frac{s^2\kappa\tau}{2}\right) + \frac{\kappa\tau s^3}{6}\left(-s\kappa \mp \frac{s^2\kappa'}{2}\right)$$

$$= \left(\frac{s^2k}{2} + \frac{k's^3}{6}\right)\left(\frac{s^2k\tau}{2}\right) - \frac{k\tau s^3}{6}\left(sk + \frac{s^2 \dots}{2}\right)$$

$$= \frac{s^2k}{2}\left(\frac{s^2k\tau}{2}\right) + \frac{k's^3}{6}\left(\frac{s^2k\tau}{2}\right) - sk\left(\frac{k\tau s^3}{6}\right) - \frac{s^2k'}{2}\left(\frac{k\tau s^3}{6}\right)$$

$$= \frac{s^4k^2\tau}{4} + \frac{s^5k'k\tau}{12} - \frac{s^4k^2\tau}{6} - \frac{s^5k'k\tau}{12}$$

$$= \frac{s^4k^2\tau}{4} - \frac{s^4k^2\tau}{6}$$

$$= \frac{6s^4k^2\tau - 4s^4k^2\tau}{24}$$

$$= \frac{s^4k^2\tau}{12}$$

$$r'(s) \times r'(0) = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 1 - \frac{s^2k^2}{2} & \frac{sk + s^2k'}{2} & \frac{s^2k\tau}{2} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \vec{t}(0) - \vec{n}\left(-\frac{s^2k\tau}{2}\right) + \vec{b}\left(\dots\right)$$

$$\vec{n}\left(\frac{s^2k\tau}{2}\right) - \vec{b}\left(\frac{sk + s^2k'}{2}\right)$$

$$|\alpha'(s) \times \alpha'(0)| = \sqrt{\left(\frac{s^2 k \tau}{2}\right)^2 + \left(sk + \frac{s^2 k'}{2}\right)^2}$$

$$= \sqrt{\frac{s^4 k^2 \tau^2}{4} + s^2 k^2 + \frac{s^4 k'^2}{4} + \frac{2 s^3 k k'}{2}}$$

$$= \sqrt{\frac{s^4 k^2 \tau^2}{4} + s^2 k^2 + \frac{s^4 k'^2}{4} + s^3 k k'}$$

$$= \sqrt{s^2 k^2}$$, omitting higher powers

Sub in (a).

$$\text{(a)} \Rightarrow d = \frac{\frac{s^4 k^2 \tau}{12}}{(s^2 k^2)^{1/2}}$$

$$= \frac{s^4 k^2 \tau}{12} \times \frac{1}{(s^2 k^2)^{1/2}}$$

$$= \frac{s^4 k^2 \tau}{12\, sk}$$

$$= \frac{s^3 k \tau}{12}$$

Curvature and Torsion of the curve given as the intersection of two surf

Theorem:

Find the curvature and torsion of the curve given as the intersection of two surfaces.

Proof:

Let the given two surfaces be

$f(x,y,z) = 0 \longrightarrow (1)$ &

$g(x,y,z) = 0 \longrightarrow (2)$. also the curve is the intersection of eqn (1) &

Let $\vec{t}$ be the unit tangent vector at the point P.

ie) t lies on the tangents of two surfaces at the point

ie) t is $\parallel^{el}$ to the common $\perp^r$ of the normals.

Now, we denote the normal of the finite surfaces are

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$$

$\vec{t}$ is the ...

Therefore $(\nabla f \times \nabla g)$ is the ...

ie) $\nabla f \times \nabla g = h$ (say).

We know that, $\vec{t}$ is $\parallel^{el}$ to common $\perp^{r}$.

ie) $\vec{t}$ is $\parallel^{el}$ to h

ie) $h = \lambda t \longrightarrow (3)$

Now, $\lambda t = \lambda \frac{d\vec{r}}{ds}$

$\lambda t = \Delta \vec{r}$, where $\Delta = \lambda \frac{d}{ds}$.

Put the operator Δ on both sides of eqn (3)

$(3) \Rightarrow \quad \lambda t = h$.

$\Rightarrow \Delta \lambda t = \Delta h$

$\Rightarrow \lambda \frac{d}{ds}(\lambda t) = \Delta h$

$\Rightarrow \lambda [\lambda' t + \lambda t'] = \Delta h$.

$\Rightarrow \lambda \lambda' t + \lambda^2 k n = \Delta h \longrightarrow (4)$

Taking the cross product of eqn (3) & (4).

$\lambda t \times (\lambda \lambda' t + \lambda^2 k n) = h \times \Delta h$

$\Rightarrow (\lambda t \times \lambda \lambda' t) + (\lambda t \times \lambda^2 k n) = h \times \Delta h$

$\Rightarrow \lambda^2 \lambda' (t \times t) + \lambda^3 k (t \times n) = h \times \Delta h$

$\Rightarrow \lambda^2 \lambda' (0) + \lambda^3 k (b) = h \times \Delta h$.

$\Rightarrow \lambda^3 k b = m \quad [\because h \times \Delta h = m]$

$\longrightarrow (5)$

Taking Mod on bothsides.

$|\lambda^3 k \vec{b}| = |m|$

$\Rightarrow \sqrt{\lambda^6 k^2} = |m|$

$\Rightarrow \lambda^3 k = |m|$.

$\Rightarrow k = \dfrac{|m|}{\lambda^3} \longrightarrow (6)$

Put the operator Δ on bothsides of eqn (5), then we have.

$\Delta \lambda^3 k \vec{B} = \Delta m$

$\Rightarrow \lambda \dfrac{d}{ds} (\lambda^3 k \vec{B}) = \Delta m$.

$\Rightarrow \lambda \left[\lambda^3 k b' + \lambda^3 k' b + 3\lambda^2 k b \lambda' \right] = \Delta m$

$\Rightarrow \lambda^4 k b' + \lambda^4 k' b + 3\lambda^3 \lambda' k b = \Delta m$

$\Rightarrow \lambda^4 k b' + \lambda^4 k' b + 3\lambda^3 \lambda' k \bar{b} = \Delta m$

$\Rightarrow \lambda^4 k (-\tau n) + \lambda^4 k' b + 3\lambda^3 \lambda' k \bar{b} = \Delta m$

$\Rightarrow -\lambda^4 k \tau \bar{n} + (\lambda^4 k' + 3\lambda^3 \lambda' k) \bar{b} = \Delta m$

$\longrightarrow (7)$

Taking the scalar product of eqn (4)

$(\lambda \lambda' t + \lambda^2 k n) \cdot [-\lambda^4 k \tau n + (\lambda^4 k' + 3\lambda^3 \lambda'$

$= \Delta h \cdot \Delta m$.

$-\lambda^6 k^2 \tau = \Delta h \cdot \Delta m$.

$-\lambda^6 \dfrac{|m|^2}{\lambda^6} \tau = \Delta h \cdot \Delta m$

$-|m|^2 \tau = \Delta h \cdot \Delta m$

$\Rightarrow -m^2\tau = \Delta h \cdot \Delta m.$

$$\Rightarrow \tau = -\frac{[\Delta h \cdot \Delta m]}{m^2}.$$

X

10 Mark

Find the curvature and torsion of the curve of intersection of two quadratic surfaces $ax^2 + by^2 + cz^2 = 1$ and $a'x^2 + b'y^2 + c'z^2 = 1$.

soln:

Given the quadratic surfaces are

$$ax^2 + by^2 + cz^2 = 1 \longrightarrow (1)$$

$$a'x^2 + b'y^2 + c'z^2 = 1 \longrightarrow (2).$$

Let us consider the two given surfaces are $f(x,y,z) = \frac{(ax^2 + by^2 + cz^2 - 1)}{2}$

and $g(x,y,z) = \frac{a'x^2 + b'y^2 + c'z^2 - 1}{2}.$

$$\Delta f = (ax + by + cz) = (ax, by, cz)$$

$$\nabla g = (a'x + b'y + c'z) = (a'x, b'y, c'z).$$

$$(\nabla f \times \nabla g) = \begin{vmatrix} \vec{i} & \vec{n} & \vec{k} \\ ax & by & cz \\ a'x & b'y & c'z \end{vmatrix}$$

$\nabla f \times \nabla g = \vec{i}\,[(by\,c'z) - b'y\,cz] -$

$\vec{j}\,[ax\,c'z - a'x\,cz] +$

$\vec{k}\,[ax\,b'y - a'x\,by]$

$= \vec{i}\,yz\,(bc' - b'c) - \vec{j}\,xz\,(ac' - a'c) +$

$\vec{k}\,xy\,(ab' - a'b).$

$= \vec{i}\,yz\,(bc' - b'c) + \vec{j}\,xz\,(a'c - ac') +$

$\vec{k}\,xy\,(ab' - a'b).$

$= \vec{i}\,yz\,(A) + \vec{j}\,xz\,(B) + \vec{k}\,xy\,(C)$

$= Ayz\,\vec{i} + Bxz\,\vec{j} + C\,xy\,\vec{k}$

$= (Ayz, Bxz, Cxy)$

$= xyz\,(A/x, B/y, C/z).$

Put $\lambda t = \cancel{x}\,\lambda\gamma' = (A/x, B/y, C/z)$ $\longrightarrow$ ③

Taking scalar product of ③ to itself

$\lambda t \cdot \lambda t = (A/x, B/y, C/z) \cdot (A/x, B/y, C/z)$

$\lambda^2(t \cdot t) = (A^2/x^2, B^2/y^2, C^2/z^2)$

$\lambda^2(1) = (A^2/x^2, B^2/y^2, C^2/z^2)$

$\lambda^2 = (A^2/x^2, B^2/y^2, C^2/z^2)$

We know that $r = (x, y, z)$

$\therefore r' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$

Put operator Δ on both sides of eqn (3).

$\Delta \lambda t = \Delta\left(A/x, B/y, C/z\right)$.

$\lambda \frac{d}{ds}(\lambda t) = \lambda\, d/ds\left(A/x, B/y, C/z\right)$.

$\lambda^2 t' + \lambda \lambda' t = \lambda\left[-\frac{A}{x^2}\frac{dx}{ds}, -\frac{B}{y^2}\frac{dy}{ds}, -C/z^2 \frac{dz}{ds}\right]$,

$\lambda^2 kn + \lambda\lambda' t = \lambda \cdot \left[-A/x^2\, dx/ds, -B/y^2 \frac{dy}{ds}, -C/z^2\, dz/ds\right]$

$\longrightarrow$ (4).

Now the eqn (3) becomes.

$\lambda\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \left(A/x, B/y, C/z\right)$.

equating, we get

$\lambda \frac{dx}{ds} = A/x$

$$\left.\begin{aligned} \Rightarrow \frac{dx}{ds} &= \frac{A}{\lambda x} \\ \frac{dy}{ds} &= \frac{B}{\lambda y} \\ \frac{dz}{ds} &= C/\lambda z \end{aligned}\right\} \longrightarrow (5)$$

Sub (5) in (4).

(4) $\Rightarrow$
$$\lambda^2 k n + \lambda\lambda' t = \lambda\left[-A/x^2 \frac{dx}{ds}, \frac{-B}{y^2}\frac{dy}{ds}, \frac{-C}{z^2}\frac{dz}{ds}\right]$$

$$\Rightarrow \lambda^2 k n + \lambda\lambda' t = \lambda\left[-A/x^2\left(\frac{A}{\lambda x}\right), \frac{-B}{y^2}\left(\frac{B}{\lambda y}\right), -C/z^2\left(C/\lambda z\right)\right]$$

$$\Rightarrow \lambda^2 k n + \lambda\lambda' t = \cancel{\lambda}\left[-\frac{A^2}{\cancel{\lambda} x^3}, -\frac{B^2}{\cancel{\lambda} y^3}, \frac{-C^2}{\cancel{\lambda} z^3}\right] \longrightarrow (6).$$

Taking the cross product of eqn (5) & (6)

$$\lambda^3 k \vec{b} = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ A/x & B/y & C/z \\ -A^2/x^3 & -B^2/y^3 & -C^2/z^3 \end{vmatrix}$$

$$= \vec{t}\left(B/y(-C^2/z^3) + B^2/y^3(C/z)\right) -$$
$$\vec{n}\left(A/x(-C^2/z^3) + A^2/x^3(C/z)\right) +$$
$$\vec{b}\left(A/x(-B^2/y^3) + A^2/x^3(B/y)\right)$$

$$= \vec{t}\left(\frac{-BC^2}{z^3 y} + \frac{B^2C}{zy^3}\right) - \vec{n}\left(-\frac{AC^2}{xz^3} + \frac{A^2C}{x^3z}\right)$$
$$\vec{b}\left(-\frac{AB^2}{xy^3} + \frac{A^2B}{x^3y}\right)$$

$$= \vec{t}\left(\frac{-BC^2}{z^3y} + \frac{B^2C}{zy^3}\right) + \vec{n}\left(\frac{-A^2C}{x^3z} + \frac{AC^2}{xz^3}\right)$$
$$\vec{b}\left(-\frac{AB^2}{xy^3} + \frac{A^2B}{x^3y}\right)$$

$$\lambda^3 k\vec{b} = \vec{t}\left[\frac{-BC^2y^2 + B^2Cz^2}{z^3y^3}\right] + \vec{n}\left[\frac{-A^2Cz^2 + AC^2x^2}{x^3z^3}\right] + \vec{b}\left[\frac{-AB^2x^2 + A^2By^2}{x^3y^3}\right].$$

$$= \vec{t}\left[\frac{BC}{zy}\left(\frac{Bz^2 - Cy^2}{y^2z^2}\right)\right] + \vec{n}\left[\frac{AC}{xz}\left(\frac{Cx^2 - Az^2}{x^2z^2}\right)\right] + \vec{b}\left[\frac{AB}{xy}\left(\frac{Ay^2 - Bx^2}{x^2y^2}\right)\right]$$

$$= \left[\frac{BC}{zy}\left(\frac{Bz^2 - Cy^2}{y^2z^2}\right), \frac{AC}{xz}\left(\frac{Cx^2 - Az^2}{x^2z^2}\right), \frac{AB}{xy}\left(\frac{Ay^2 - Bx^2}{x^2y^2}\right)\right] \longrightarrow (7)$$

eqn (1) $\times a'$ $-$ (2) $\times a$ $\Rightarrow$.

$$aa'x^2 + a'by^2 + a'cz^2 = a'$$
$$aa'x^2 + ab'y^2 + ac'z^2 = a.$$
$$(-) \quad (-) \quad (-)$$

$$(a'by^2 - ab'y^2) + (a'cz^2 - ac'z^2) = a' - a.$$

$$\Rightarrow (a'b - ab')y^2 + (a'c - ac')z^2 = a' - a.$$

$$\Rightarrow -Cy^2 + Bz^2 = a' - a.$$

$$Cy^2 - Bz^2 = a - a' \longrightarrow (8)$$

(Since $A = bc' - b'c$
$B = a'c - ac'$
$C = ab' - a'b$.)

① $\times b'$ − ② $\times b$.

$\Rightarrow ab'x^2 + \cancel{bb'y^2} + b'cz^2 = b'$

$\Rightarrow a'bx^2 + \cancel{bb'y^2} + bc'z^2 = b$.

(−) (−) (−)

$(ab' - a'b)x^2 + (b'c - bc')z^2 = b' - b$.

$\Rightarrow Cx^2 - Az^2 = b' - b$. $\longrightarrow$ (8b)

① $\times c'$ − ② $\times c$

$\Rightarrow ac'x^2 + bc'y^2 + \cancel{cc'z^2} = c'$

$\Rightarrow a'cx^2 + b'cy^2 + \cancel{cc'z^2} = c$

(−) (−) (−)

$(ac' - a'c)x^2 + (bc' - b'c)y^2 = c' - c$

$\Rightarrow -Bx^2 + Ay^2 = c' - c$

$\Rightarrow Bx^2 - Ay^2 = c - c'$. $\longrightarrow$ (8c)

Sub 8a, 8b, 8c in eqn (7)

$$(7) \Rightarrow \left[\frac{BC}{zy}\left(\frac{Bz^2 - Cy^2}{y^2 z^2}\right), \frac{AC}{xz}\left(\frac{Cx^2 - Az^2}{x^2 z^2}\right), \frac{AB}{xy}\left(\frac{Ay^2 - Bx^2}{x^2 y^2}\right)\right].$$

$$\Rightarrow \left[\frac{BC}{zy}\left(\frac{a'-a}{y^2 z^2}\right), \frac{AC}{xz}\left(\frac{b'-b}{x^2 z^2}\right), \frac{AB}{xy}\left(\frac{c'-c}{x^2 y^2}\right)\right].$$

$$= \frac{ABC}{x^3 y^3 z^3}\left[\frac{x^3(a'-a)}{A}, \frac{y^3(b'-b)}{B}, \frac{z^3(c'-c)}{C}\right] \longrightarrow (9)$$

$$\mu \bar{b} = \left[\frac{x^3(a'-a)}{A}, \frac{y^3(b'-b)}{B}, \frac{z^3(c'-c)}{C}\right] \longrightarrow (10)$$

where $\mu = \lambda^3 k$

Taking scalar product of (10) with (10)

we get

$$\mu^2 = \frac{x^6(a'-a)^2}{A^2} + \frac{y^6(b'-b)^2}{B^2} + \frac{z^6(c'-c)^2}{C^2}$$

$$\mu^2 = \sum \frac{x^6(a'-a)^2}{A^2}$$

Taking scalar product of (9) with (9)

$$\lambda^6 k^2 = \frac{A^2B^2C^2}{x^6 y^6 z^6} \sum \frac{x^6(a'-a)^2}{A^2} \quad \text{—— (11)}$$

$$\lambda^6 k^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\mu^2 \longrightarrow (12)$$

$$\lambda^3 k = \frac{ABC}{x^3y^3z^3}\mu$$

$$\Rightarrow \mu = \frac{\lambda^3 k\, x^3y^3z^3}{ABC} \longrightarrow (13)$$

Differentiate (10) w.r.to 's'

$$\mu b' + \mu' b = \left[\left(\frac{a'-a}{A}\right)3x^2\frac{dx}{ds}, \left(\frac{b'-b}{B}\right)3y^2\frac{dy}{ds}, \left(\frac{c'-c}{C}\right)3z^2\frac{dz}{ds}\right]$$

$$\mu(-\tau\bar{n}) + b\mu' = 3\left[\left(\frac{a'-a}{A}\right)\frac{x^2A}{\lambda x}, \left(\frac{b'-b}{B}\right)\frac{y^2B}{\lambda y}, \left(\frac{c'-c}{C}\right)\frac{z^2C}{\lambda C}\right]$$

$$-\mu\tau\bar{n} + b\mu' = 3/\lambda\left[x(a'-a), y(b'-b), z(c'-c)\right]$$

$$\Rightarrow -\lambda\mu\tau\bar{n} + \lambda\mu'\bar{b} = 3\left[x(a'-a), y(b'-b), z(c'-c)\right] \longrightarrow (14)$$

$$(11) \Rightarrow \lambda^6 k^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\sum\frac{x^6(a'-a)^2}{A^2}$$

$$= \frac{A^2B^2C^2}{x^6y^6z^6}\sum\frac{x^6(a'-a)^2}{A^2}$$

$$\lambda^{8} = \left(\frac{A^2B^2C^2}{x^6y^6z^6} \sum \frac{x^6(a'-a)^2}{A^2}\right) \Big/ \left(\sum \frac{A^2}{x^2}\right)^3$$

Since $\lambda^2 = \sum \frac{A^2}{x^2}$ from (3).

Take the scalar product of (5) & (4)

we get.

$$(\lambda^2 k\bar{D} + \lambda\lambda' \bar{E})(-\lambda\mu\tau\bar{D} + \lambda b\mu') =$$

$$-\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right) . 3\left\{x(a'-a), y(b'-b), z(c'-c)\right\}$$

$$-\lambda^3 k\mu\tau = -3\left\{\frac{A^2}{x^2}(a'-a), \frac{B^2}{y^2}(b'-b), \frac{C^2}{z^2}(c'-c)\right\}$$

$$\Rightarrow \tau = \frac{3\sum \frac{A^2}{x^2}(a'-a)}{\lambda^3 k\mu}.$$

$$= \frac{3\sum \frac{A^2}{x^2}(a'-a)}{\lambda^3 . k \frac{\lambda^3 k x^3y^3z^3}{ABC}}$$

$$= \frac{3AC\sum \frac{A^2}{x^2}(a'-a)}{\lambda^6 x^3y^3z^3\left[\frac{A^2B^2C^2}{x^6y^6z^6} \sum \frac{x^6}{A^2}(a'-a)^2\right]}$$

$$= \frac{3x^3y^3z^3 \sum \frac{A^2}{x^2}(a'-a)}{ABC\sum \frac{x^6}{A^2}(a'-a)^2}$$

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Defn:-

The surface is the locus of a point $p(x,y,z)$ satisfying some restriction which is expressed by a relation of the form. $f(x,y,z)=0$. This called the implicit or constraint equation.

Note:-

1. Implicit equations of a surface is convanient for the study of algebraic surfaces are a whole.

2. for regions which are not too large. explicit form, in which the coordinates o and a form on the surface are expressed interms of two parameters is prefarable.

Defn:-

The parametric or freedom eqns of a surface are of the form $x=f(u,v)$. $y=g(u,v)$ $z=h(u,v)$, where u and v are parameters which take real values and vary freely in some domain D and the functions f, g, h are single valued continuous and posses continuous partial derivatives of the r^{th} order.

In this case, the surface is said to be of class r. The parameters u and v are called curvillinear coordinates

Defn:

Let there to be two parametric representation u, v and u', v' of the same surface.

Any transformation of the form $u' = \phi(u,v)$ and $v' = \psi(u,v)$ relating these two representations is called a parametric transformation.

A parametric transformation is said to be proper if.

(i) ϕ and ψ are single valued functions and

(ii) $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in some domain D. $\begin{vmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} \neq 0$

ie) ϕ and ψ have non vanishing Jacobian.

Defn:. ordinary point.

Let $\bar{r} = (x,y,z)$ be the position vector of a point P on the surface.

Since x,y,z are continuous functions of parameters u,v possessing partial derivatives of required order, we can take $\bar{r} = \bar{r}(u,v)$ as the parametric eqn of the surface.

partial differentiation w.r to u & v will be denoted by suffixes 1 and 2 respectively so that. $\bar{r_1} = \frac{\partial \bar{r}}{\partial u}$ and $\bar{r_2} = \frac{\partial \bar{r}}{\partial v}$ P is called an ordinary point if $\bar{r_1} \times \bar{r_2} \neq 0$ at P.

ie) If rank $\begin{pmatrix} \\ x_2 & y_2 & z_2 \end{pmatrix} = 2$.

p is called a singular point if

$\bar{r}_1 \times \bar{r}_2 = 0$ at p.

ie) If rank $\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \neq 2$.

$\therefore$ rank is either 0 or 1.

Curves on a surface

Defn:

Let $\bar{r} = \bar{r}(u,v)$ be the equation of a surface of class r, defined on a domain D. Let $u = u(t)$, $v = v(t)$ be a curve of class s lying in D.

Then $\bar{r} = \bar{r}(u(t), v(t))$ is a curve lying on the surface with the class equal to the smaller of r and s the eqns, $u = u(t)$, $v = v(t)$ are called as the curvilinear equation of the curve.

Def: parametric curve.

Let $\bar{r} = \bar{r}(u,v)$ be the equation of a surface of class r. defined on a domain D. Let v have the constant value c.

Then as u varies, the point $\bar{r} = \bar{r}(u, c)$ traces a curve called the parametric curve $v = c$.

Note 1:

There is one parametric curve for every value of c; together they from the parametric curves $v = $ constant.

Note 2:-

There is one parametric curve for every value of c. together they form the system of parametric curves u-constant.

Note:3

Through every point of the surface, there passes one and only one parametric curve of each system.

Note:4.

No two parametric curves of a system intersect.

Defn:- orthogonal.

The two parametric curves through a point p are orthogonal. If $\bar{r_1} \cdot \bar{r_2} = 0$ at p. If this condition is satisfied at every point ie $\forall$ u,v in the domain D. The two systems of parametric curves are orthogonal

Defn:- Note:-

consider a surface with no singularities for the general curve given by $u = u(t)$, $v = v(t)$ the tangent is in the direction.

$$\frac{d\bar{r}}{dt} = \frac{\partial \bar{r}}{\partial u} \cdot \frac{du}{dt} + \frac{\partial \bar{r}}{\partial v} \cdot \frac{dv}{dt} = \bar{r_1} \frac{du}{dt} + r_2 \cdot \frac{dv}{dt}.$$

since $\bar{r_1}$ and r_2 are non zero and independent, the tangent to the curves on the surface through a point lie in the plane which contains the two vectors $\bar{r_1}$ and $\bar{r_2}$ at p. This plane is the tangent plane at p.

Defn:-

The normal to the surface at p is the normal to the tangent plane at p and is therefore $\perp^r$ to $\bar{r_1}$ and $\bar{r_2}$. The sense of the normal is fixed by the condition that if $\hat{N}$ is the unit normal vector, then $\bar{r_1}, \bar{r_2}$ and $\bar{N}$ in this order form a right-handed system.

It follows that,

(unit normal vector) $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\hat{N} = \frac{\bar{r_1} \times \bar{r_2}}{H}, \quad H = |\bar{r_1} \times \bar{r_2}| \neq 0.$$

Remark:-

parametric eqn of the surface is not unique.

Ex:-

consider the constraint eqn of the surface,

$$x^2 - y^2 = z.$$

Let $x = u,\ y = v,\ z = u^2 - v^2$

now,

$$x^2 - y^2 = u^2 - v^2 = z.$$

$x = u+v,\ y = u-v$ and $z = 4uv$.

now, $x^2 - y^2 = (u+v)^2 - (u-v)^2$

$$= u^2 + v^2 + 2uv - u^2 - v^2 + 2uv$$

$$= 4uv$$

$$= z.$$

hence ① & ② are the parametric eqn represe the surface $x^2 - y^2 = z$.

Remark:-

Parametric eqn, and constraint equivalent.

Example:-

Let us consider the eqn $x = u\cosh v,\ y = u\sinh v,$

$z = u^2 \longrightarrow ①$.

Sol.

$x^2 - y^2 = u^2\cosh^2 v - u^2\sinh^2 v.$

$= u^2 = z.$

Hence $x = u\cosh v,\ y = u\sinh v\ \ z = u^2$ is the

Parametric eqn, of the constraint eqn,

$x^2 - y^2 = z$ of the surface.

The eqn ② represent whole of the paraboloid.

But ① represent only part of the surface.

for which $z \geq 0$, since u takes only real

values.

Note:-

The Jacobian matrix defined by.

$$\frac{\partial(\phi,\psi)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial\phi}{\partial u} & \frac{\partial\psi}{\partial u} \\ \frac{\partial\phi}{\partial v} & \frac{\partial\psi}{\partial v} \end{vmatrix} = \begin{vmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{vmatrix}$$

Theorem 21.

The property of an ordinary point is unaltered by a proper parameter transformation (or)

In a proper parameter " of the singular point transformed into regular point.

Proof:-

Let u, v and u', v' be two set of parameters and $u' = \phi(u,v)$ and $v' = \psi(u,v)$.

Then, we have.

$$\vec{r_1} = \frac{\partial\vec{r}}{\partial u} = \frac{\partial\vec{r}}{\partial u'} \cdot \frac{\partial u'}{\partial u} + \frac{\partial\vec{r}}{\partial v'} \cdot \frac{\partial v'}{\partial u}$$

$$= \frac{\partial \vec{r}}{\partial u'} \cdot \frac{\partial \phi}{\partial u} + \frac{\partial \vec{r}}{\partial v'} \cdot \frac{\partial \psi}{\partial u} \quad \text{(1)}$$

$$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = \frac{\partial \vec{r}}{\partial u'} \cdot \frac{\partial u'}{\partial v} + \frac{\partial \vec{r}}{\partial v'} \cdot \frac{\partial v'}{\partial v}.$$

$$= \frac{\partial \vec{r}}{\partial u'} \cdot \frac{\partial \phi}{\partial v} + \frac{\partial \vec{r}}{\partial v'} \cdot \frac{\partial \psi}{\partial v} \quad \text{(2)}$$

(1) x (2) give,

$$\vec{r_1} \times \vec{r_2} = \left(\frac{\partial \vec{r}}{\partial u'} \cdot \frac{\partial \phi}{\partial u} + \frac{\partial \vec{r}}{\partial v'} \cdot \frac{\partial \psi}{\partial u} \right) \times \left(\frac{\partial \vec{r}}{\partial u'} \cdot \frac{\partial \phi}{\partial v} + \frac{\partial \vec{r}}{\partial v'} \cdot \frac{\partial \psi}{\partial v} \right)$$

$$= \frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial u'} \right) + \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} \left(\frac{\partial \vec{r}}{\partial v'} \times \frac{\partial \vec{r}}{\partial v'} \right)$$

$$\frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \right) + \frac{\partial \psi}{\partial u} \cdot \frac{\partial \phi}{\partial v} \left(\frac{\partial \vec{r}}{\partial v'} \times \frac{\partial \vec{r}}{\partial u'} \right).$$

$$= 0 - \frac{\partial \psi}{\partial u} \cdot \frac{\partial \phi}{\partial v} \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \right) + \frac{\partial \phi}{\partial u} \cdot \frac{\partial \psi}{\partial v} \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \right).$$

$$= \frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \left(\frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial v} - \frac{\partial \psi}{\partial u} \cdot \frac{\partial \psi}{\partial v} \right).$$

$$\vec{r_1} \times \vec{r_2} = \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \right) \frac{\partial(\phi, \psi)}{\partial(u, v)}$$

$$\therefore \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) = \frac{\partial(\phi, \psi)}{\partial(u, v)} \left(\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \right).$$

Since the transformation is proper.

$$\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0.$$

Hence, If $\vec{r_1} \times \vec{r_2} \neq 0$, then $\frac{\partial \vec{r}}{\partial u'} \times \frac{\partial \vec{r}}{\partial v'} \neq 0$

Hence the regular point transformed into singularity.

Types of singularity :-

A paint which is not an ordinary paint is called a singularity.

Types :-

Essential singularities

Artificial singulaities.

Essential singularities :-

are due to particular, geometrical features of the surfaces and are independent of the choice of parametric representation.

Ex :-

A vertex of cone.

Artificial singularities :-

An artificial singulaities arise from the choice of a particular parametric representation.

Ex :-

The origin of the polar coordinates, $\vec{r} = (u\cos v, u\sin v, 0)$ in the plane $u=0$.

1. find the eqn of tangent plane to the surface.

Let the eqn of the surface be $\vec{r} = \vec{r}(u,v)$. then.

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{du} \cdot \frac{du}{ds} + \frac{d\vec{r}}{dv} \cdot \frac{dv}{ds}.$$

$$\vec{t} = \vec{r_1} \cdot \frac{du}{ds} + \vec{r_2} \cdot \frac{dv}{ds}.$$

① $\Rightarrow \vec{t} = \vec{r_2} \cdot \frac{dv}{ds}$.

① $\Rightarrow \vec{t}$ is parallel to $\vec{r_2}$.

for the curve $v=$ constant $\cdot \frac{dv}{ds} = 0$.

① $\Rightarrow \vec{t} = \vec{r_1} \cdot \frac{du}{ds}$.

2). cartesian eqn of the tangent plane to the surface.

Sol.

Let $\vec{R}$ be a position vector of the point p. on the plane. then the eqn of the tangent plane is $(\vec{R} - \vec{r})$ grad $f = 0$.

where grad $f = \nabla \cdot f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$.

Let $\vec{R} = X, Y, Z$.

$\vec{r} = (x, y, z)$.

① becomes, $(X-x, Y-y, Z-z) \cdot \left(\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial z} \right) = 0$.

$(X-x) \frac{\partial f}{\partial x} + (Y-y) \frac{\partial f}{\partial y} + (Z-z) \frac{\partial f}{\partial z} = 0. \rightarrow ②$

$\frac{\partial f}{\partial u} = 0 \Rightarrow \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u} = 0. \rightarrow ③$

And $\frac{\partial f}{\partial v} = 0 \quad \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial v} = 0. \rightarrow ④$

elemenate . $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ from ② ③ ④.

we get,
$$\begin{vmatrix} X-x & Y-y & Z-z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{vmatrix} = 0 \text{ //.}$$

Surface of revolution.

Defn:-

A surface generated by the rotation of a plane curve about an axis in its plane is called a surface of revolution.

Defn: pitch.

The distance translated in one complete revolution called pitch of the helicoid is the constant $2\pi a$.

It is +ve or -ve according as the helicoid is right or left handed and is zero for the surface of revolution.

Problem: 1.

If $\vec{r} = [g(u)\cos v, g(u)\sin v, f(u)]$ find $\bar{N}$.

Sol).

Given $\vec{r} = (g(u)\cos v, g(u)\sin v, f(u))$.

$$\vec{r_1} = \frac{\partial \vec{r}}{\partial u} = (g'(u)\cos v, g'(u)\sin v, f'(u)).$$

$$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (-g(u)\sin v, g(u)\cos v, 0).$$

$$\vec{r_1}\cdot\vec{r_2} = -g(u)g'(u)\sin v\cos v + g(u)g'(u)\sin v\cos v = 0$$

$$\vec{r_1}\cdot\vec{r_2} = 0.$$

The parametric curves are orthogonal.

To find $\vec{N}$ (direction of Normal).

$$N = \frac{\vec{r_1}\times\vec{r_2}}{|\vec{r_1}\times\vec{r_2}|}.$$

$\vec{r_1}\times\vec{r_2} =$

$= \vec{i}\,(0 - g$

$\vec{r_1}\times\vec{r_2} = -$

$|\vec{r_1}\times\vec{r_2}|^2 =$

$|\vec{r_1}\times\vec{r_2}| =$

$N =$

$N =$

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point

~~Helicoid~~

screw

Defn:.

The

not onl

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g'(u)\cos v & g'(u)\sin v & f'(u) \\ -g(u)\sin v & g(u)\cos v & 0 \end{vmatrix}$$

$$= \vec{i}\,(0 - g(u) f'(u)\cos v) - \vec{j}\,(0 + g(u) f'(u)\sin v)$$
$$+ \vec{k}\,(g'(u) g(u)\cos^2 v + g'(u) g(u)\sin^2 v).$$
$$g(u)\,[g'(u)\cos^2 v + g'(u)\sin^2 v]$$

$$\vec{r_1} \times \vec{r_2} = -g(u)\,(f'(u)\cos v,\ f'(u)\sin v,\ -g'(u)).$$

$$|\vec{r_1} \times \vec{r_2}|^2 = (g(u))^2\,(f'(u)^2\cos^2 v + (f'(u))^2\sin^2 v + (g'(u))^2$$
$$f'(u)^2(\cos^2 v + \sin^2 v)$$
$$= (g(u))^2\,((f'(u))^2 + (g'(u))^2.$$

$$|\vec{r_1} \times \vec{r_2}| = g(u)\,((f'(u))^2 + (g'(u))^2)^{1/2}.$$

$$N = \frac{-g(u)\,[f'(u)\cos v,\ f'(u)\sin v,\ -g'(u)]}{g(u)\,((f'(u))^2 + (g'(u))^2)^{1/2}}$$

$$N = \frac{-1}{\sqrt{f'(u)^2 + (g'(u))^2}}\,(f'(u)\cos v,\ f'(u)\sin v,\ -g'(u)).$$

The anchor ring.

The anchor ring is the surface generated circle of radius 'a' about a line in its plane and at a distance $b > a$ from its centre.

Note: The position vector of a point on the anchor ring is

$$\vec{r} = ((b + a\cos u)\cos v,\ (b + a\cos u)\sin v,\ a\sin u)$$

~~Helicoids~~ :-

screw motion

Defn:.

There are surfaces which are obtained by not only by a rotation buy by a rotation with translation such a motion is called.

Screw motion.

Defn :- Helicoid.

The surface generated by the screw motion of a curve about a fixed line called the axis is called helicoid.

Right helicoid :-

Right helicoid is the helicoid generated by a straight line which meets the axis at the right angle.

General helicoid :-

2m. The general helicoid with z-axis as the axis is generated by the curve of intersection of the surface with any plane containing the z-axis.

Metric. (or) First Fundamental Form.

Defn :-

On a given surface $\bar{r} = \bar{r}(u,v)$. consider a curve defined by $u = u(t)$, $v = v(t)$.

Then $\bar{r}$ is a function of t along the curve.

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{d\bar{r}}{dt}\right)^2 = \left(\frac{\partial \bar{r}}{\partial u}\cdot\frac{du}{dt} + \frac{\partial \bar{r}}{\partial v}\cdot\frac{dv}{dt}\right)^2.$$

$$= \left(\frac{\partial \bar{r}}{\partial u}\right)^2\left(\frac{du}{dt}\right)^2 + 2\left(\frac{\partial \bar{r}}{\partial u}\cdot\frac{du}{dt}\right)\left(\frac{\partial \bar{r}}{\partial v}\cdot\frac{dv}{dt}\right) + \left(\frac{\partial \bar{r}}{\partial v}\right)^2\left(\frac{dv}{dt}\right)^2.$$

$$= \bar{r}_1^{\,2}\left(\frac{du}{dt}\right)^2 + 2\cdot\frac{\partial \bar{r}}{\partial u}\cdot\frac{\partial \bar{r}}{\partial v}\cdot\frac{du}{dt}\cdot\frac{dv}{dt} + \bar{r}_2^{\,2}\left(\frac{dv}{dt}\right)^2.$$

$$= \bar{r}_1^2 \left(\frac{du}{dt}\right)^2 + 2\bar{r}_1 \bar{r}_2 \frac{du}{dt} \cdot \frac{dv}{dt} + \bar{r}_2^2 \left(\frac{dv}{dt}\right)^2.$$

$$ds^2 = \bar{r}_1^2 du^2 + 2\bar{r}_1 \cdot \bar{r}_2 \cdot du\,dv + \bar{r}_2^2 \cdot dv^2.$$

first fundamental form

$$ds^2 = E du^2 + 2F du dv + G dv^2.$$

where $E = \bar{r}_1^2 \quad F = \bar{r}_1 \cdot \bar{r}_2 \cdot G = \bar{r}_2^2.$

$E, F, G \Rightarrow I^{st}$ fundamental co-efficient

Relation between E, F, G and H.

$E = \bar{r}_1^2, \; F = \bar{r}_1 \cdot \bar{r}_2 \quad G = \bar{r}_2^2 ; \; H = |\bar{r}_1 \times \bar{r}_2|.$

$$EG - F^2 = \bar{r}_1^2 \bar{r}_2^2 - (\bar{r}_1 \bar{r}_2)^2.$$

$$= \bar{r}_1^2 \bar{r}_2^2 - \bar{r}_1^2 \bar{r}_2^2 \cos^2\theta.$$

$$= \bar{r}_1^2 \bar{r}_2^2 \cdot (1 - \cos^2\theta).$$

$$= \bar{r}_1^2 \bar{r}_2^2 \sin^2\theta.$$

$$= |\bar{r}_1 \times \bar{r}_2|^2$$

$$= H^2.$$

$$\boxed{EG - F^2 = H^2.}$$

Angle between parametric curves.

Let P be the point of intersection of the parameter curves. Let $\bar{r}$ be its position vector. Then $\bar{r}_1 = \frac{\partial \bar{r}}{\partial u}$ and $\bar{r}_2 = \frac{\partial \bar{r}}{\partial v}$ are along the tangents to the parametric curves, v = constant and u = constant.

Let $w \; (0 < w < \pi)$ be the angle between the parameter curves.

$\vec{a} \cdot \vec{b} = \vec{a}\vec{b} \cos\theta$

$\vec{a} \times \vec{b} = \vec{a}\vec{b} \sin\theta$

Then, $\sin w = \frac{|\bar{r}_1 \times \bar{r}_2|}{|\bar{r}_1| \, |\bar{r}_2|}$ and.

$$\cos\omega = \frac{\bar{r}_1 \cdot \bar{r}_2}{|\bar{r}_1||\bar{r}_2|}.$$

[$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$, $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{N}$.

$$|\bar{a}\times\bar{b}| = ||\bar{a}||\bar{b}|\sin\theta\,\hat{N}|$$
$$= |\bar{a}||\bar{b}|\sin\theta.$$

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$$\tan\omega = \frac{\sin\omega}{\cos\omega}$$
$$= \frac{|\bar{r}_1\times\bar{r}_2|}{\bar{r}_1\cdot\bar{r}_2} = \frac{H}{F}.$$

$$\omega = \tan^{-1}\left(\frac{H}{F}\right)$$

Problem:-

(7) The necessary and sufficient condition for the parametric curves on a surface are orthogonal iff $F=0$.

Sol.

suppose that the parametric curves are orthogonal

ie) $\theta = 90^\circ$.

w.k.t, the angle between the parametric curve is $\tan\theta = \frac{H}{F}$.

$$\tan 90^\circ = \frac{H}{F} \Rightarrow F = 0.$$

conversely, suppose $F=0$.

$$\tan\theta = \frac{H}{F} = \frac{H}{0}.$$

$$\tan\theta = \infty \Rightarrow \theta = \tan^{-1}\infty$$
$$\theta = 90^\circ.$$

The parametric curves are orthogonal.

Theorem:-

The metric

transformat

Sol

let u,v

and $u' = \phi$

Now,

Hence th

Paramet

Note:

E, F, G

transfo

Theorem:-

The metric is invarient under a parametric transformation.

10M

Sol

Let u,v and u',v' be two set of parameters. and $u' = \phi(u,v)$ and $v' = \varphi(u,v)$.

Now,

$$E'du'^2 + 2F'du'dv' + G'dv'^2 = \vec{r_1}'^2 du'^2 + 2(\vec{r_1}', \vec{r_2}').du'.dv' + (\vec{r_2}'^2 \, dv'^2).$$

$$= (\vec{r_1}' du' + \vec{r_2}' dv')^2$$

$$= \left\{ \left(\frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial u'} \right) du' + \left(\frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial v'} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial v'} \right) dv' \right\}^2$$

$$= \left\{ \vec{r_1} \left(\frac{\partial u}{\partial u'} + \vec{r_2} \frac{\partial v}{\partial u'} \right) du' + \left(\vec{r_1} \frac{\partial u}{\partial v'} + \vec{r_2} \frac{\partial v}{\partial v'} \right) dv' \right\}^2$$

$$= \left[\vec{r_1} \left(\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv' \right) + \vec{r_2} \left(\frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv' \right) \right]^2$$

$$= (\vec{r_1} du + \vec{r_2} dv)^2$$

$$= (\vec{r_1})^2 du^2 + \vec{r_2}^2 dv^2 + 2\vec{r_1}\vec{r_2} du dv$$

$$= E du^2 + 2F du dv + G dv^2.$$

Hence the metric is invarient under a parametric transformation.

Note:-

The 1^{st} fundamental coefficients. E, F, G are not invarient under a parametric transformation.

Problem:-

Show that the curves bisecting the angles between the parametric curves are given by

$E du^2 - G dv^2 = 0.$

proof:-

φ is the angle between any two parametric curves (du, dv) and $(\partial u, \partial v)$ then,

$$\cos\varphi = \frac{E du \partial u + F(du \partial u + dv \partial u) + G dv \partial v}{ds \partial s}.$$

If φ_1 is the angle between the parametric curve v = constant and bisecting curves of direction (du, dv).

since v = constant $\partial v = 0$.

$$\partial\vec{r} = \vec{r_1}\,\partial u.$$

$$\partial s = |\partial\vec{r}| = \sqrt{\vec{r_1}^2 \partial u^2} = \sqrt{E \partial u^2}.$$

$$\partial s = \sqrt{E}\,\partial u$$

$$\cos\varphi_1 = \frac{E du \partial u + F dv \partial u}{\sqrt{E}\, ds\, \partial u} \longrightarrow (1).$$

$|||^{ly}$, φ_2 is the angle between the parametric curve u = constant and bisecting curves, of direction (du, dv).

since u = constant, $\partial u = 0$.

$$\partial\vec{r} = \vec{r_2}\,\partial v.$$

$$\partial s = |\partial\vec{r}| = \sqrt{\vec{r_2}^2 \partial v^2} = \sqrt{G dv^2} = \sqrt{G}\,.\,\partial v.$$

$$\cos\varphi_2 = \frac{F du \partial v + G dv \partial v}{\sqrt{G}\, \partial v\, ds} \longrightarrow (2).$$

Since the curves bisecting the angles, between the parametric curves $\varphi = \varphi_2$, and $F = 0$.

$\cos\varphi_1 = \cos\varphi_2.$

$$\Rightarrow \frac{E\,du\,\partial u + F\,dv\,\partial u}{\sqrt{E}\,ds\,\partial u} = \frac{F\,du\,\partial v + G\,dv\,\partial v}{\sqrt{G}\,\partial v\,ds}.$$

$$\frac{E\,du\,\partial u}{\sqrt{E}\,\partial u} = \frac{G\,dv\,\partial v}{\sqrt{G}\,\partial v} \quad [F=0].$$

$$\sqrt{E}\,du = \sqrt{G}\,dv.$$

$$E\,\partial u^2 = G\,\partial v^2$$

$$\Rightarrow E\,\partial u^2 - G\,\partial v^2 = 0.$$

* Find a surface of revolution which is isometric with a region of the right helicoid.

Sol

The surface of revolution is given by,

$$\vec{r} = (g(u)\cos v,\ g(u)\sin v,\ f(u)).$$

for the fn f and g.

$$\vec{r_1} = (g\cos v,\ g\sin v,\ f(u)).$$

$$\vec{r_2} = (-g\sin v,\ g\cos v,\ 0).$$

$$E = \vec{r_1}^2 = g_1^2 + f_1^2.$$

$F' = 0$ and $G' = u'^2 + a^2.$

It's metric, $ds'^2 = E'du'^2 + 2F'du'dv' + G'dv'^2$

$$ds'^2 = du'^2 + (u'^2 + a^2)\,dv'^2.$$

Find the transformation (u, v) to (u', v') which makes these two metric identical.

Sol

Let $v' = v$ and $u' = \phi(u).$

Then, $dv' = dv$ & $du' = \phi_1 du.$

The metric are identical if

$g^2 = u'^2 + a^2$ and $g_1^2 + f_1^2 = \phi_1^2$.

i.e.) $g^2 = \phi^2 + a^2 \longrightarrow ①$ & $g_1^2 + f_1^2 = \phi_1^2$.

put $\phi(u) = a\sinh u$ and $g(u) = a\cosh u$.

Then ① satisfies,

$$② \Rightarrow g_1^2 + f_1^2 = \phi_1^2.$$

$$a^2\sinh^2 u + f_1^2 = a^2\cosh^2 u.$$

$$\Rightarrow f_1 = 0$$

$$\Rightarrow f = au.$$

Hence the right helicoid is isometric with the surface obtained by revolving the curve.

$$x = a\cosh u.$$

$$y = 0.$$

$$z = au \Rightarrow u = \frac{z}{a}.$$

The generating curve is the catenary. $x = a\cosh(z/a)$ with the parametric a and directrix the z axis and the surface of revolution is catenoid.

Problem: 1.

find the paraboloid $x = u,\ y = v,\ z = u^2 - v^2$, find E, F, G and H. [Find the First Fundamental coefficient]

Sol

$$\vec{r} = \vec{r}(x, y, z) = \vec{r}(u, v, u^2 - v^2).$$

$$\vec{r_1} = \frac{\partial \vec{r}}{\partial u} = (1, 0, 2u).$$

$$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (0, 1, -2v).$$

$$E = \vec{r_1}^2 = 1^2 + 0^2 + (2u)^2 = 1 + 4u^2.$$

$E = 1+4u^2.$

$F = \vec{r_1}\cdot\vec{r_2} = (1.0.2u)(0,1,-2v).$

$= 0+0-4uv.$

$F = -4uv.$

$G = \vec{r_2}^2 = 0^2+1^2+(-2v)^2$

$= 1+4v^2.$

$G = 1+4v^2.$

$(a+b\cos u)\cos v, (a+b\cos u)\sin v, a\sin u.$

$H = \sqrt{EG-F^2}$

$= \sqrt{(1+4u^2)(1+4v^2)-(-4uv)^2}$

$= \sqrt{1+4u^2+4v^2+16u^2v^2-16u^2v^2}$

$H = \sqrt{1+4u^2+4v^2}.$

Problem : 2.

calculate the first fundamental coefficients and the area of the anchor ring corresponding to the domain $0 \le u \le 2\pi,\ 0 \le v \le 2\pi.$ $4\pi^2 ab.$

Sol.

The position vector of any point on the anchor ring is.

$\vec{r} = ((b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u)$

$\vec{r_1} = \frac{\partial \vec{r}}{\partial u} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (-(b+a)\cos u)\sin v, (b+a\cos u)\cos v, 0).$

$E = \vec{r_1}^2;\ F = \vec{r_1}\cdot\vec{r_2};\ G = \vec{r_2}^2;\ H = \sqrt{EG-F^2}.$

$E = \vec{r_1}^2 = a^2\sin^2 u\cos^2 v + a^2\sin^2 u\sin^2 v + a^2\cos^2 u.$

$= a^2\sin^2 u + a^2\cos^2 u.$

$E = a^2$

$F = \vec{r_1}\cdot\vec{r_2} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u).$
$(-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0).$

$= a(b+a\cos u)\sin u\sin v\cos v - a(b+a\cos u)$
$\sin u\sin v\cos v + 0,$

$F = 0.$

$G = \vec{r_2}^{\,2} = (b+a\cos u)^2\sin^2 v + (b+a\cos u)^2\cos^2 u.$

$G = (b+a\cos u)^2.$

$H = \sqrt{EG - F^2}.$

$= \sqrt{a^2(b+a\cos u)^2 - 0}$

$H = a(b+a\cos u).$

The first fundamental coefficients are

$E = a^2,\ F = 0,\ G = (b+a\cos u)^2.$

The elementary area ds for the surface is given by $ds = H\,du\,dv.$

$H = \sqrt{EG - F^2} = a(b+a\cos u).$

$\therefore$ The area of the anchor ring corresponding to the domain $0 \le u \le 2\pi$ $0 \le v \le 2\pi$ is.

$$S = \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\,du.dv.$$

$$= a\int_0^{2\pi} dv \int_0^{2\pi} (b+a)\cos u)\,du.$$

$$= a[v]_0^{2\pi}\int_0^{2\pi}(b+a\cos u).du.$$

$$= a[v]_0^{2\pi}[bu + a\sin u]_0^{2\pi}$$

$$= a(2\pi - 0)[b(2\pi) + a\sin 2\pi - 0].$$

$$= 2\pi a(2\pi b + a\times 0)$$

$$= 2\pi a(2\pi b)$$

$S = 4\pi^2 ab.$

Direction coefficients

$\vec{a} \cdot \hat{N}$

$a_n =$

-Defn:-

At a point p of a surface, there are three independent vectors $\hat{N}$, $\vec{r_1}$ and $\vec{r_2}$. Therefore, every vector $\vec{a}$ at p can be expressed in the form $\vec{a} = a_n\hat{N} + \lambda\vec{r_1} + \mu\vec{r_2}$ where the scalars a_n, λ and μ are defined uniquely by this relation. This gives $\vec{a}$ as the sum of two vectors $a_n\hat{N}$ normal to the surface and $\lambda\vec{r_1} + \mu\vec{r_2}$ in the tangent plane at p.

The scalar a_n is called the normal component of $\bar{a}$ and is given by $a_n = \vec{a} \cdot \hat{N}$.

The vector $\lambda\vec{r_1} + \mu\vec{r_2}$ is called the tangential part of $\vec{a}$ and λ, μ are the tangential components of $\vec{a}$.

Note:-

1) The vector $\vec{a} = a_n\hat{N} + \lambda\vec{r_1} + \mu\vec{r_2}$ lies in the tangent plane iff $a_n = 0$.

2) The vector $\vec{a} = a_n\hat{N} + \lambda\vec{r_1} + \mu\vec{r_2}$ is normal to the surface iff $\lambda = \mu = 0$.

3) If $\vec{a} = (\lambda, \mu)$ Then $\vec{\alpha} = \lambda\vec{r_1} + \mu\vec{r_2}$ and so.

$$|a| = |\lambda\vec{r_1} + \mu\vec{r_2}| = \sqrt{\lambda^2\vec{r_1}^2 + \mu\vec{r_2}^2 + 2\lambda\mu\vec{r_1}\cdot\vec{r_2}}.$$

$$= \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}.$$

$\therefore$ The magnitude of a tangential vector (λ, μ) is $\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}$.

Def :- direction coefficients.

A direction in the tangent plane at p on a surface is conveniently described by the components of the unit vector in this direction. These components are called direction coefficients and are written as (l, m).

Note :-

since the vector (l, m) has unit magnitude the coefficients satisfy.

$$\therefore El^2 + 2Flm + Gm^2 = 1.$$

Angle between two directions :-

Let (l, m) and (l', m') be the direction coefficients of two directions at p.

Then $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$= \frac{H(lm' - l'm)}{Ell' + F(lm' + l'm) + Gmm'}.$$

Note :-

condition for orthogonal direction.

$$Ell' + F(lm' + l'm) + Gmm' = 0.$$

Def :- direction ratio.

Two scalars λ, μ which are proportional respectively to l, m. The direction coefficients of a direction are called direction ratio of the direction.

Relation between direction ratio and direction coefficients

Let (l, m) and (λ, μ) be the direction

coefficients
of a direct
Then $(l, m$
Note :-
conditi
terms of
$E\lambda\lambda' + F$
Problem 1 :-
Find
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Sol
Let $(l'$
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Then si
cos
ie) $H(l$
$Ell' +$
② can
$l'($
$\Rightarrow l'$
$\Rightarrow$
$\Rightarrow$

coefficients and direction ratio's respectively of a direction.

Then $(l, m) = \dfrac{(\lambda, \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$

Note:-

condition for orthogonal direction in terms of direction ratios is

$E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0.$

Problem 1.:-

Find the coefficients of the direction which makes the angle $\pi/2$ with the direction whose coefficients are (l, m).

Sol.

Let (l', m') be the coefficient of the direction which makes the angle $\pi/2$ with the direction whose coefficients are (l, m).

Then $\sin\frac{\pi}{2} = H(lm' - l'm)$ and

$\cos\frac{\pi}{2} = Ell' + F(lm' + l'm) + Gmm'.$

ie) $H(lm' - l'm) = 1 \longrightarrow ①$ and

$Ell' + F(lm' + l'm) + Gmm' = 0. \longrightarrow ②$

② can be written as, $Ell' + Flm' + Fl'm + Gmm'$

$l'(El + Fm) + (Fl + Gm)m'$

$l'(El + Fm) + m'(Fl + Gm) = 0$

$\Rightarrow l'(El + Fm) = -m'(Fl + Gm).$ $\quad l'(El + Fm) = -m'(Fl + Gm)$

$\Rightarrow \dfrac{l'}{Fl + Gm} = \dfrac{-m'}{El + Fm} = k\text{(say)}.$ $\quad \dfrac{l'}{Fl + Gm} = \dfrac{-m'}{El + Fm}$

$\Rightarrow l' = k(\cancel{El + F}\cdot Fl + Gm)$ and. $\quad l' = k(Fl + Gm)$

$m' = -k(El + Fm). \longrightarrow ③$

sub these in ① we get.

$H(-lK(El+Fm) - Km(Fl+Gm)) = 1.$

$\Rightarrow KH(-El^2 - Flm - Flm - Gm^2) = 1.$

$\Rightarrow -KH(El^2 + 2Flm + Gm^2) = 1.$

$\Rightarrow -KH = 1.$ [$\because$ (l,m) are d.c, $El^2 + 2Flm + Gm^2 = 1$]

$\Rightarrow K = \frac{-1}{H}.$

$\therefore$ from ③.

$l' = -\frac{1}{H}(Fl+Gm)$ and.

$m' = \frac{1}{H}(El+Fm).$

$(l', m') = \left[\frac{-1}{H}(Fl+Gm), \frac{1}{H}(El+Fm)\right]$ are.

required coefficients of the direction.

Problem 2:-

A surface of revolution is defined by the eqn. $x = \cos u \cos v$, $y = \cos u \sin v$, $z = -\sin u + \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right)$. S.T the metric is. $\tan^2 u\, du^2 + \cos^2 u\, dv^2$.

Sol

$\vec{r} = \left(\cos u \cos v, \cos u \sin v, -\sin u + \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right)\right)$

we know that,

$ds^2 = E du^2 + 2F du dv + G dv^2.$

where $E = \vec{r_1}^2$ $F = \vec{r_1} \cdot \vec{r_2}$ $G = \vec{r_2}^2$

Now, $\vec{r_1} = \frac{\partial \vec{r}}{\partial u}.$

$= \left(-\sin u \cos v, -\sin u \sin v, -\cos u + \frac{1}{\tan\left(\frac{\pi}{4} + \frac{u}{2}\right)} \times \sec^2\left(\frac{\pi}{4} + \frac{u}{2}\right) \times \frac{1}{2}\right)$

$\vec{r_1} = \left(-\sin u \cos v, -\sin u \sin v, -\cos u + \frac{1}{\cos u}\right)$ -2

$(\sin 2\theta = 2\sin\theta \cos\theta;$

$\sin 2\left(\frac{\pi}{4} + \frac{u}{2}\right) = \sin\left(\frac{\pi}{2} + u\right) = \cos u)$

$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (-\cos u \sin v, \cos u \cos v, 0)$

$E = \vec{r_1}^2 = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u + \frac{1}{\cos^2 u} - 2.$

$= \sin^2 u + \cos^2 u + \frac{1}{\cos^2 u} - 2$

$= 1 + \sec^2 u - 2.$

$= \sec^2 u - 1$

$E = \tan^2 u.$

$F = \vec{r_1} \cdot \vec{r_2} = \sin u \sin v \cos u \cos v - \sin u \sin v \cos u \cos v + 0$

$F = 0.$

$G = \vec{r_2}^2 = \cos^2 u \sin^2 v + \cos^2 u \cos^2 v + 0$

$G = \cos^2 u.$ $E du^2 + 2F du dv + G dv^2$

$ds^2 = \tan^2 u \, du^2 + \cos^2 u \, dv^2$

Family of curves.

Defn:-

Family of curves on a surface is a system given by an implicit equation. $\phi(u,v) = c$, where ϕ is single valued and has continuous derivatives ϕ_1, ϕ_2, which do not vanish together and c is a real parameter.

$\frac{1}{\sin 2(\pi/4 + u/2)} = \frac{1}{\sin(\pi/2 + u)} = \frac{1}{\cos u}$

Theorem:-

The restriction ϕ_1 and ϕ_2 should not vanish together at any point on ϕ is desirable (must).

proof:-

For, let $\phi(u,v) = c$ be the given family of curves.

Then $d\phi = 0$.

$\Rightarrow \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} \cdot dv = 0$.

$\Rightarrow \phi_1 du + \phi_2 dv = 0$.

$\Rightarrow \phi_1 du = -\phi_2 dv$.

$\Rightarrow \frac{du}{-\phi_2} = \frac{dv}{\phi_1}$.

$Edu^2 + Gdv^2 + Fdudv$.

$Edudu + F(dudv + dvdu) + Gdvdv$.

$(-\phi_2, \phi_1)$ are direction ratios of the tangent at (u,v) to the curve $\phi(u,v) = c$ which passes through the point (u,v).

If at any point, both ϕ_1 and ϕ_2 vanish together, then the directions are indeterminate and so we do not have a definite tangent at the point the above restriction on ϕ is desirable.

Defn:- orthogonal trajectories:-

Let $\phi(u,v) = c$ be the given family of curves lying on the surface $\vec{r} = \vec{r}(u,v)$.

If there exist another family of curves $\psi(u,v) = c'$ lying on the same surface has two curves, one from each family are orthogonal, then the curves of the second family are called orthogonal trajectories of the first family.

Problem:

On the parabalid $x^2 y^2 = z$. find the orthogonal trajectories of the sections by the planes z = constant.

Soln:

consider the parabolid $x^2 y^2 = z$

the parametric representation of the surface can be taken as $x = u$, $y = v$, $z = u^2 - v^2$,

The position vector of a point on the surface is $\vec{r}(u,v) = (u, v, u^2 - v^2)$.

consider the section of the parabolid by the plane z = constant.

Then $\phi(u,v) = u^2 - v^2$ = constant $\longrightarrow$ (1) is the first family of curves.

$$\phi_1 = \frac{\partial \phi}{\partial u} = 2u, \quad \phi_2 = \frac{\partial \phi}{\partial v} = -2v.$$

The tangential direction at any point on the surface is $(-\phi_2, \phi_1) = (2v, 2u)$ or we can take it as (v, u).

let (du, dv) be direction ratious of the orthogonal direction at (u,v).

Then, by the condition of orthogonality. we have,

$$E v\,du + F(u\,du + v\,dv) + G u\,dv = 0 \longrightarrow (2).$$

Now,

$$\vec{r} = (u, v, u^2 - v^2).$$

$$\vec{r_1} = \frac{\partial \vec{r}}{\partial u} = (1, 0, 2u).$$

$$\vec{r_2} = \frac{\partial \vec{r}}{\partial v} = (0, 1, -2v).$$

$F = \vec{r_1} \cdot \vec{r_2} = -4uv.$

$G = \vec{r_2}^{\,2} = 1+4v^2.$

(2) becomes,

$(1+4u^2)\,v\,du \cdot 4uv\,(u\,du + v\,dv) + (1+4v^2)\,u\,dv = 0.$

$(1+4u^2)\,v\,du \cdot 4uv\,(u\,du+v\,dv) + (-4uv^2 + u + 4uv^2)\,dv = 0.$

$\Rightarrow (v+4u^2v - 4u^2v)\,du + (-4uv^2 + u + 4uv^2)\,dv = 0.$

$\Rightarrow v\,du + u\,dv = 0.$

$\Rightarrow d(uv) = 0.$

$\Rightarrow uv = \text{constant}.$

$\therefore$ The orthogonal trajectories is $uv = \text{constant}.$

Double family of curves.

If P, Q and R are continuous functions of u and v, which do not vanish together, and $Q^2 - PR > 0$, then the quadratic differential eqn.

$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ has for solns two families of curves.

Theorem.

The two directions given by $P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ are orthogonal on a surface iff $ER - 2QF + GP = 0.$

proof:-

Let (λ, μ) and (λ', μ') be direction ratio's of the two families of curves given by,

$Pdu^2 + 2Qdudv + Rdv^2 = 0.$

Then, $\frac{\lambda}{\mu}$ and $\frac{\lambda'}{\mu'}$ are the roots of the eqn

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0.$$

$$\therefore \frac{\lambda}{\mu} + \frac{\lambda'}{\mu'} = -\frac{2Q}{P} \text{ and } \frac{\lambda}{\mu}\cdot\frac{\lambda'}{\mu'} = \frac{R}{P}. \longrightarrow ①$$

W.K.T

the directions (λ, μ) and (λ', μ') are orthogonal

iff $E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0.$

i.e) iff $E\frac{\lambda}{\mu}\frac{\lambda'}{\mu'} + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G = 0$

i.e) iff $E\frac{R}{P} - \frac{2Q}{P}F + G = 0$

i.e iff $ER - 2QF + GP = 0.$

$\therefore$ The condition for the two families of curves to be orthogonal is.

$$ER - 2FQ + GP = 0.$$

Note:-

consider the eqn $Pdu^2 + 2Qdudv + Rdv^2 = 0. \longrightarrow ①$

where P and Q are continuous functions of u and v which do not vanish together.

If $P = R = 0$ then ① reduces to $dudv = 0.$

$\Longrightarrow du = 0$ (or) $dv = 0.$

$\Longrightarrow u = \text{constant}$ (or) $v = \text{constant}.$

$\therefore$ ① gives two families of parametric curves ; the condition for orthogonality if $F = 0.$

In the condition $E - 2FQ + GP = 0$, take $P = R = 0$

Problem:

Prove that if θ is the angle at the point (u,v) between the two directions given by $P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$. Then, $\tan\theta = \dfrac{2H(Q^2 - RP)^{1/2}}{ER - 2FQ + GP}$

proof:

Let (λ, μ) and (λ', μ') be direction ratios of the two families of curves given by.

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0.$$

Then, $\frac{\lambda}{\mu}$ & $\frac{\lambda'}{\mu'}$ are the roots of the eqn.

$$P\left(\frac{du}{dv}\right)^2 + 2Q\,\frac{du}{dv} + R = 0.$$

$$\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right)^2 - 4\left(\frac{\lambda}{\mu}\,\frac{\lambda'}{\mu'}\right)$$

$$\therefore \frac{\lambda}{\mu} + \frac{\lambda'}{\mu'} = -\frac{2Q}{P} \;\&\; \frac{\lambda}{\mu}\,\frac{\lambda'}{\mu'} = \frac{R}{P} \longrightarrow (1).$$

Now,

$$\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right)^2 = \left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right)^2 - 4\,\frac{\lambda}{\mu}\cdot\frac{\lambda'}{\mu'}.$$

$$= \frac{4Q^2}{P^2} - \frac{4R}{P}.$$

$$= \frac{4(Q^2 - PR)}{P^2}$$

$$\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'} = \frac{2\sqrt{Q^2 - PR}}{P}$$

Let θ be the angle between the two directions. then,

$$\tan\theta = \frac{H(\lambda\mu' - \lambda'\mu)}{E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu'}$$

$$= \frac{H\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right)}{E\frac{\lambda}{\mu}\cdot\frac{\lambda'}{\mu'} + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G}$$

$$= \frac{2H\sqrt{Q^2-PR}}{P\left(\frac{ER}{P} - \frac{2FQ}{P} + G\right)}$$

$$= \frac{2H\sqrt{Q^2-PR}}{ER-2FQ+GP}$$

$$\tan\theta = \frac{2H(Q^2-PR)^{1/2}}{ER-2FQ+GP}$$

Problem

Show that on a right helicoid, the family of curves orthogonal to $u\cos v$ = constant is the family $(u^2+a^2)\sin^2 v$ = constant.

Soln

Equation of a right helicoid is

$\vec{r} = (u\cos v, u\sin v, av)$ where a is a constant.

$\vec{r_1} = \frac{\partial\vec{r}}{\partial u} = (\cos v, \sin v, 0)$ and

$\vec{r_2} = \frac{\partial\vec{r}}{\partial v} = (-u\sin v, u\cos v, a)$.

$\therefore E = \vec{r_1}^2 = 1$, $F = \vec{r_1}\cdot\vec{r_2} = 0$, $G = \vec{r_2}^2 = u^2+a^2$.

given family of curves is $u\cos v$ = constant. $\longrightarrow (1)$

diff (1) we get, $u(-\sin v)dv + \cos v\,du$

$u(-\sin v)dv + (\cos v)du = 0$.

$\cos v\, du = u \sin v\, dv$.

$\Rightarrow \dfrac{du}{u\sin v} = \dfrac{dv}{\cos v}$.

$\therefore$ direction ratio's of the tangent to the given family $u\cos v = $ constant are $(u\sin v, \cos v)$.

Let (du, dv) be the direction orthogonal to the given family.

Then by condition of orthogonality.

$u\sin v\, du + (u^2+a^2)\cos v\, dv = 0$.

$\Rightarrow u\sin v\, du = -(u^2+a^2)\cos v\, dv$.

$\Rightarrow \dfrac{u\,du}{u^2+a^2} = -\dfrac{\cos v}{\sin v}\, dv$.

$u^2+a^2 = t$.
$2u\,du = dt$.

$\Rightarrow \int \dfrac{u\,du}{u^2+a^2} = -\int \dfrac{\cos v}{\sin v}\, dv$.

$\Rightarrow \frac{1}{2}\log(u^2+a^2) = -\log \sin v + \log c$.

$\Rightarrow \log(u^2+a^2) = -2\log\sin v + 2\log c$.

$\Rightarrow \log(u^2+a^2) + \log\sin^2 v = 2\log c$.

$\Rightarrow \log(u^2+a^2)\sin^2 v = \log c^2$

$\Rightarrow (u^2+a^2)\sin^2 v = c^2$

$\Rightarrow (u^2+a^2)\sin^2 v = $ constant.

$\therefore$ The required family of curves is.

$(u^2+a^2)\sin^2 v = $ constant //.

UNIT - III

Geodesics

Defn:-

Let A and B be two given points on a surfaces S. Let these points be joined by curves lying on S. Then any curve possessing stationary length for small variation over S is called a geodesic.

Geodesic differential equation:-

Let A and B be two points on a surface $\vec{r} = \vec{r}(u,v)$ on the surface let us consider all the arcs joining A and B given parametrically as $u = u(t)$, $v = v(t)$ where $u(t), v(t)$ are of class 2.

for every arc α. Let us assume that $t = 0$ at A and $t = 1$ at B so that the arcs are defined on $[0,1]$.

Let α be one such arc and $s(\alpha)$ be the length of the arc α joining A and B on the surfaces.

W.K.T $ds^2 = E du^2 + 2F du\,dv + G dv^2$.

$$\dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

$$s(\alpha) = \int_0^1 \dot{s}\,dt \qquad \left[\dot{s} = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\right.$$

$$= \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\,dt$$

Keeping the end points A and B fixed,

Let us deform the [illegible] to obtain α. Then we can take the equation of the new arc α' parametrically as.

$u'_t = u(t) + \varepsilon\lambda(t)$, $v'(t) = v(t) + \varepsilon\mu(t)$.

where $\varepsilon > 0$, is small and $\lambda(t)$, $\mu(t)$ are such that.

$\lambda(0) = \mu(0) = 0$ and $\lambda(1) = \mu(1) = 0$.

Let the length of the arc be $s(\alpha')$ then.

$$s(\alpha') = \int_0^1 \sqrt{E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'}\, dt'.$$

The variation in $s(\alpha)$ is $s(\alpha') - s(\alpha)$ and of order ε.

Defn:- stationary

If α is such that the variation in $s(\alpha)$ is atmost of order ε^2 for all small variations in α (ie, $\forall\ \lambda(t)$ and $\mu(t)$).

Then $s(\alpha)$ is said to be stationary and α is a geodesic.

Equation for geodesics as in the calculus of variation:.

Let $f = \sqrt{2T}$. where $T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}(E\dot{u} + 2F\dot{u}\dot{v} + G\dot{v}^2)$.

Then $s(\alpha') - s(\alpha) = \left|\int_0^1 [\, f(u+\varepsilon\lambda, v+\varepsilon\mu, \dot{u}+\varepsilon\dot{\lambda}, \dot{v}+\varepsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})]\, dt.\right|$

By Taylor's theorem for several variables we've,

$$= \varepsilon\int_0 (\lambda \frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} + \dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}})\, dt$$

$$u = \frac{\partial f}{\partial u}\frac{du}{dt} = \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right). \quad = \int_0^1 \frac{\partial f}{\partial \dot u}\, d\lambda = \left[\lambda \frac{\partial f}{\partial \dot u}\right]_0^1 - \int_0^1 \lambda\, d\left(\frac{\partial f}{\partial \dot u}\right)$$

$dv = \lambda$

$v = \lambda.$

$$= \lambda\left[\frac{\partial f}{\partial \dot u}\right]_0^1 - \int_0^1 \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) dt$$

The first term on the R.H.S is zero because $\lambda = 0$ at $t=0$ and $t=1$.

$$-\int_0^1 \dot\lambda \frac{\partial f}{\partial \dot u}\, dt = -\int_0^1 \lambda \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right). dt$$

Semeliarly,

$$\int_0^1 \dot\mu \frac{\partial f}{\partial \dot v}\, dt = -\int_0^1 \mu \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right). dt.$$

$$S(\alpha') - S(\alpha) = \varepsilon \int_0^1 \left(\lambda \frac{\partial f}{\partial u} + \mu \frac{\partial f}{\partial v} - \lambda \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) - \mu \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right)\right.$$

$$\left. dt + O(\varepsilon^2)\right).$$

$$S(\alpha') - S(\alpha) = \varepsilon \int_0^1 (\lambda L + \mu M)\, dt + O(\varepsilon^2)$$

where

$$\left.\begin{aligned} L &= \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) \\ M &= \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right) \end{aligned}\right\} \longrightarrow (1).$$

$\therefore S(\alpha)$ is stationary and so α is a geodesic.

iff $u(t)$ and $v(t)$ are such that

$$\int (\lambda L + \mu M). dt = 0 \longrightarrow (2)$$

for all admissible ... which satisfy ...

class & in $0 \le t \le 1$. which satisfy

at $t=0$ and $t=1$.

It will now be proved that this

condition implies $L = M = 0$

Lemma :-

If $g(t)$ is continuous for $0 < t < 1$, and if

$\int_0^1 v(t)\, g(t)\, dt = 0$ for all admissible functions

$v(t)$ as defined above, then $g(t) = 0$.

proof:-

suppose there is a t_0 between 0 and 1

such that $g(t_0) \neq 0$, say $g(t_0) > 0$.

Then, since g is continuous $g(t) > 0$ in

some interval (a, b) where $0 < a < t_0 < b < 1$.

Now, we define $v(t)$ as follows.

$v(t) = 0$ for $0 \le t \le a$ and for $b < t \le 1$ and.

$v(t) = (t-a)^3 (b-t)^3$ for $a \le t \le b$.

Then, $v(t)$ is admissible and.

$$\int_0^1 v(t)\, g(t)\, dt = \int_a^b v(t)\, g(t)\, dt > 0.$$

since $g > 0$ and $v > 0$. for $a < t < b$. The

supposition that there is a t_0 such that

$g(t_0) \neq 0$ is false.

This is a contradiction.

Hence $g(t) = 0$.

Hence the lemma is proved.

Geodesic equation:-

The function L and M in equation ① are continuous because, E, F, G are assumed to be of class 1 and u(t), v(t). of class 2. The lemma can therefore be applied to equation ② first with $\mu = 0$ and λ, L in place of v, g and then with $\lambda = 0$ and μ, M in plane of v, g.

It follows that equation ② is satisfied for all admissible functions λ, μ iff $L = M = 0$.

$\mu = 0$, $\lambda L\ v\ g$, $\lambda = 0$, μM.

ie) iff $L = M = 0$.

ie) iff $\left| \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) = 0 \right.$ and, $\left. \right|$

$$\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) = 0.$$

iff $\frac{\partial}{\partial u}\sqrt{2T} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{u}}\sqrt{2T}\right) = 0$ and,

$$\frac{\partial}{\partial v}\sqrt{2T} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{v}}\sqrt{2T}\right) = 0.$$

iff $\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial \dot{u}}\right) = 0$ and,

$$\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial v} - \frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial \dot{v}}\right) = 0$$

iff $\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial u} - \frac{1}{\sqrt{2T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial \dot{u}}\frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\right) = 0.$

and, $\frac{1}{\ }\cdot\frac{\partial T}{\ } - \frac{1}{\ }\frac{d}{\ }\left(\frac{\partial T}{\ }\right) - \frac{\partial T}{\ }\frac{d}{\ }\left(\frac{1}{\ }\right) = 0.$

iff

$$\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial u} - \frac{1}{\sqrt{2T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) + \frac{\partial T}{\partial \dot u}\left(-\frac{1}{(2T)^{3/2}}\right)\frac{dT}{dt} = 0.$$

and,

$$\frac{1}{\sqrt{2T}}\cdot\frac{\partial T}{\partial v} - \frac{1}{\sqrt{2T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) + \frac{\partial T}{\partial \dot v}\cdot\frac{1}{(2T)^{3/2}}\frac{dT}{dt} = 0.$$

iff,

$$\frac{1}{2T}\cdot\frac{\partial T}{\partial \dot u}\cdot\frac{dT}{dt} = \frac{d}{dt}\frac{\partial T}{\partial \dot u} - \frac{\partial T}{\partial u} \text{ and.}$$

$$\frac{1}{2T}\cdot\frac{\partial T}{\partial \dot v}\cdot\frac{dT}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v}.$$

∴ The geodesic equations are,

$$U = \frac{d}{dt}\frac{\partial T}{\partial \dot u} - \frac{\partial T}{\partial u} = \frac{1}{2T}\cdot\frac{\partial T}{\partial \dot u}\cdot\frac{dT}{dt} \text{ and.}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\,\frac{\partial T}{\partial \dot v}\cdot\frac{dT}{dt}.$$

where $T = T(u, v, \dot u, \dot v) = \frac{1}{2}[E\dot u^2 + 2F\dot u\dot v + G\dot v^2]$

and the left hand members of the equations are denoted by U and V for convenience.

Theorem:

Let $T = T(u, v, \dot u, \dot v) = \frac{1}{2}[E\dot u^2 + 2F\dot u\dot v + G\dot v^2]$.

Let $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) - \frac{\partial T}{\partial u}$, and $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v}$.

Then, $\dot u U + \dot v V = \frac{dT}{dt}$.

proof:

$T = \frac{1}{2}[E\dot u^2 + 2F\dot u\dot v + G\dot v^2]$ is a homogeneous function of $\dot{u}$ and $\dot{v}$ of degree 2.

By Euler's Theorem,

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T \longrightarrow ①.$$

since $T = T(u, v, \dot{u}, \dot{v})$.

$$dT = \frac{\partial T}{\partial u}du + \frac{\partial T}{\partial v}dv + \frac{\partial T}{\partial \dot{u}}d\dot{u} + \frac{\partial T}{\partial \dot{v}}d\dot{v}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v}.$$

Again,

$$\frac{dT}{dt} = 2\frac{dT}{dt} - \frac{dT}{dt}$$

$$= \frac{d}{dt}(2T) - \frac{dT}{dt}$$

$$= \frac{d}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right) - \left(\frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v}\right).$$

$$= \dot{u}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) + \ddot{u}\frac{\partial T}{\partial \dot{u}} + \frac{\partial T}{\partial \dot{v}}\ddot{v} + \dot{v}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)$$

$$+ \frac{\partial T}{\partial \dot{v}}\ddot{v} - \frac{\partial T}{\partial u}\dot{u} - \frac{\partial T}{\partial v}\dot{v} - \frac{\partial T}{\partial \dot{u}}\ddot{u} - \frac{\partial T}{\partial \dot{v}}\ddot{v}.$$

$$= \dot{u}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}\right] + \dot{v}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}\right].$$

$$= \dot{u}U + \dot{v}V.$$

$$\dot{u}U + \dot{v}V = \frac{dT}{dt}.$$

Hence the theorem.

Theorem 2.

If $T = T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right]$.

and $U = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}$ and $V = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}$,

Then, $\dot{u}U = \dot{v}V = \frac{dT}{dt}$.

proof:-

$T = \left[\frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)\right]$ is a homogeneous functions of $\dot{u}$ and $\dot{v}$ of degree 2.

By Euler's Theorem.

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T \longrightarrow (1)$$

$$U = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}, \quad V = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}.$$

$$\dot{u}U + \dot{v}V = \dot{u}\left(\frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}\right) + \dot{v}\left(\frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}\right).$$

$$= \frac{1}{2T}\cdot\frac{dT}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right)$$

$$= \frac{1}{2T}\frac{dT}{dt}\,2T \quad (\text{by } (1)).$$

$$\dot{u}U + \dot{v}V = \frac{dT}{dt}.$$

Theorem 3.

A necessary and sufficient condition for a curve $u = u(t)$ $v = v(t)$ on a surface $\vec{r} = \vec{r}(u, v)$ to be a geodesic is that.

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0 \text{ where, } U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} \text{ and}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}.$$

proof:

Let $f = \sqrt{2T}$ where $T = T(u, v, \dot{u}, \dot{v})$.

(equation for geodesics).

iff $\frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{u}} \cdot \frac{dT}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} \longrightarrow (1)$ and.

$\frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{v}} \frac{dT}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} \longrightarrow (2)$

iff.

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{u}} \cdot \frac{dT}{dt}.$$

and, $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial \dot{v}} = \frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{v}} \cdot \frac{dT}{dt}.$

where $T = T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right]$

and ~~ae~~ ~~do~~ left hand members of the equations are denoted by U and V for convenience.

Next, we prove that,

$$U = \frac{d}{dt} \cdot \left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{u}} \cdot \frac{dT}{dt} \text{ and}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{v}} \cdot \frac{dT}{dt}.$$

iff, $U \frac{\partial T}{\partial \dot{v}} - V \frac{\partial T}{\partial \dot{u}} = 0.$

Assume that.

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T} \cdot \frac{\partial T}{\partial \dot{u}} \cdot \frac{dT}{dt} \text{ and.}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T} \frac{\partial T}{\partial \dot{v}} \cdot \frac{dT}{dt}.$$

Then,

$$\frac{U}{\frac{\partial T}{\partial \dot{u}}} = \frac{1}{2T}\cdot\frac{dT}{dt} = \frac{V}{\frac{\partial T}{\partial \dot{v}}}.$$

$$\Rightarrow U\frac{\partial T}{\partial \dot{v}} = V\frac{\partial T}{\partial \dot{u}}.$$

$$\Rightarrow U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0.$$

Then, $U\frac{\partial T}{\partial \dot{v}} = V\frac{\partial T}{\partial \dot{u}}$.

$$\Rightarrow \frac{U}{\frac{\partial T}{\partial \dot{u}}} = \frac{V}{\frac{\partial T}{\partial \dot{v}}} = Q \text{ (say).}$$

$$\Rightarrow U = Q\cdot\frac{\partial T}{\partial \dot{u}} \text{ and } V = Q\cdot\frac{\partial T}{\partial \dot{v}} \longrightarrow (5)$$

$$U\dot{u} + V\dot{v} = Q\left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right) \longrightarrow (6)$$

since $T = T(u, v, \dot{u}, \dot{v})$ is a homogeneous equation of degree 2 in $\dot{u}$ and $\dot{v}$.

By Euler's Theorem,

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T.$$

(6) becomes, $U\dot{u} + V\dot{v} = 2TQ \longrightarrow (7)$

claim :-

The two expressions of U in (4) satisfy the equation.

$$\dot{u}U + \dot{v}V = \frac{dT}{dt}$$

for,

$T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$ is a homogeneous $\dot{u}U + \dot{v}V = \frac{dT}{dt}$ write form.

previous thm.

considering the 2nd expressions of U and V.

$$\dot{u}U + \dot{v}V = \dot{u}\left(\frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}\right) + \dot{v}\left(\frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}\right).$$

..... write from thm.

$$\therefore \dot{u}U + \dot{v}V = \frac{dT}{dt}.$$

Hence the claim is proved.

① becomes.

$$U = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} \text{ and } V = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}.$$

$$\therefore U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} \text{ and.}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}.$$

Thus we have proved,

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} \text{ and.}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \text{ iff.}$$

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0 \longrightarrow ⑧$$

combining ③ and ⑧ we get,

The curve α is a geodesic iff

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$$

problem: 1.

P.T a necessary and sufficient condition for a curve $v=c$ on the general surface to be a geodesic is $EE_2+E_1F-2EF_1=0$.

U.Q proof:

W.K.T a necessary and sufficient condition for a curve $u=u(t)$ and $v=v(t)$ to be a geodesic is $U\frac{\partial T}{\partial \dot v}-V\frac{\partial T}{\partial \dot u}=0 \longrightarrow (1)$.

where $U=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)-\frac{\partial T}{\partial u} \longrightarrow (2)$.

$V=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right)-\frac{\partial T}{\partial v}$.

and $T=\frac{1}{2}(E\dot u^2+2F\dot u\dot v+G\dot v^2) \longrightarrow (3)$

from (3),

$\frac{\partial T}{\partial \dot u}=\frac{1}{2}(2E\dot u+2F\dot v)=E\dot u+F\dot v \longrightarrow (A)$

[$\because$ E, F, G are functions of u, v].

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)=E\ddot u+\dot E\dot u+F\ddot v+\dot F\dot v$.

$=E\ddot u+(E_1\dot u+E_2\dot v)\dot u+F\ddot v+(F_1\dot u+F_2\dot v)\dot v$.

[$\dot E=\frac{dE}{dt}=\frac{\partial E}{\partial u}\cdot\frac{du}{dt}+\frac{\partial E}{\partial v}\cdot\frac{dv}{dt}=E_1\dot u+E_2\dot v$].

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)=E\ddot u+E_1\dot u^2+(E_2+F_1)\dot u\dot v+F_2\dot v^2+F\ddot v \longrightarrow (4)$

$\frac{\partial T}{\partial u}=\frac{1}{2}(E_1\dot u^2+2F_1\dot u\dot v+G_1\dot v^2) \longrightarrow (5)$.

$$\frac{\partial T}{\partial \dot v} = \frac{1}{2}(2F\dot u + 2G\dot v) = F\dot u + G\dot v \longrightarrow (**)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = F\ddot u + \dot F\dot u + G\ddot v + \dot G\dot v.$$

$$= F\ddot u + (F_1\dot u + F_2\dot v)\dot u + (G_1\dot u + G_2\dot v)\dot v + G\ddot v.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = F\ddot u + F_1\dot u^2 + (F_2+G_1)\dot u\dot v + G_2\dot v^2 + G\ddot v \longrightarrow (6)$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}(E_2\dot u^2 + 2F_2\dot u\dot v + G_2\dot v^2) \longrightarrow (7)$$

now the given curve is $v=c$. on this curve, u can be taken as parameter.

$\therefore$ The given curve is.

$u=t,\ v=c;$

$\dot u = 1,\ \dot v = 0;$

$\ddot u = 0,\ \ddot v = 0;$

(4) (5) (6) (7) becomes.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) = E_1; \quad \frac{\partial T}{\partial u} = \frac{E_1}{2};$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = F_1; \quad \frac{\partial T}{\partial v} = \frac{E_2}{2},$$

from (2),

$$U = E_1 - \frac{E_1}{2} = \frac{E_1}{2} \text{ and } V = F_1 - \frac{E_2}{2}.$$

also, from * and **.

$$\frac{\partial T}{\partial \dot u} = E \text{ and } \frac{\partial T}{\partial \dot v} = F.$$

(1) becomes.

$$\frac{E_1}{2}F - \left(F_1 - \frac{E_2}{2}\right)E = 0.$$

$\Leftrightarrow$ i.e) $\frac{E_1}{2} F - F_1 E + \frac{E_2 E}{2} = 0.$

$\Leftrightarrow$ i.e) $E_1 E_2 + E_1 F - 2EF_1 = 0.$

$\therefore$ The necessary and sufficient condition for a curve $v=0$ on the general surface to be a geodesic is $EE_2 + E_1 F - 2EF_1 = 0$ //.

Problem 2

prove that $u=c$ is geodesic iff $GG_1 + FG_2 - 2GF_2 = 0.$

proof:-

W.K.T. a necessary and sufficient condition for a curve $u=u(t)$ and $v=v(t)$ to be a geodesic is $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0. \longrightarrow (1)$

where $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u},$

$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} \longrightarrow (2).$

and, $T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2) \longrightarrow (3)$

from (3).

$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}(2E\dot{u} + 2F\dot{v}) = E\dot{u} + F\dot{v} \longrightarrow (4).$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = E\ddot{u} + \dot{E}\dot{u} + F\ddot{v} + \dot{F}\dot{v}$

$= E\ddot{u} + (E_1\dot{u} + E_2\dot{v})\dot{u} + F\ddot{v} + (F_1\dot{u} + F_2\dot{v})\dot{v} \longrightarrow (5).$

$\frac{\partial T}{\partial u} = \frac{1}{2}(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2) \longrightarrow (6).$

$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}(F\dot{u} + 2G\dot{v}) = F\dot{u} + G\dot{v} \longrightarrow (7).$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = F\ddot u + \dot F\dot u + G\ddot v + \dot G\dot v.$$

$$= F\ddot u + (F_1\dot u + F_2\dot v)\dot u + (G_1\dot u + G_2\dot v)\dot v + G\ddot v$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = F\ddot u + F_1\dot u^2 + (F_2+G_1)\dot u\dot v + G_2\dot v^2 + G\ddot v \longrightarrow (8)$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}(E_2\dot u^2 + 2F_2\dot u\dot v + G_2\dot v^2) \longrightarrow (9).$$

now, the given curve is $u=c$. on this curve, v can be taken as parameter.

$\therefore$ The given curve is $u=c,\ v=t.$

$$\dot u = 0,\ \dot v = 1$$

$$\ddot u = 0,\ \ddot v = 0.$$

(5) (6) (8) and (9) becomes,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) = F_2;\quad \frac{\partial T}{\partial u} = \frac{G_1}{2}.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) = G_2;\quad \frac{\partial T}{\partial v} = \frac{G_2}{2}.$$

from (2).

$$U = F_2 - \frac{G_1}{2} \text{ and } V = G_2 - \frac{G_2}{2} = \frac{2G_2 - G_2}{2} = \frac{G_2}{2}.$$

also from (4) and (7).

$$\frac{\partial T}{\partial \dot u} = F \text{ and } \frac{\partial T}{\partial \dot v} = G.$$

$\therefore$ (1) becomes,

$$\left(F_2 - \frac{G_1}{2}\right)G - \left(\frac{G_2}{2}\right)F = 0.$$

i.e.) $F_2G - \frac{G_1}{2}G - \frac{1}{2}G_2F = 0.$

i.e.) $2F_2G - G_1G - G_2F = 0.$

i.e.) $GG_1 + FG_2 - 2GF_2 = 0.$

$\therefore$ The $u=c$ is geodesic iff $GG_1 + FG_2 - 2GF_2 = 0$.

Problem 3.

Prove that the curves of the family $\frac{v^3}{u^2}$ = constant are geodesics on a surface with the metric $v^2du^2 - 2uv\,du\,dv + 2u^2dv^2$, $u>0, v>0$.

proof:

W.K.T a necessary and sufficient condition for a curve $u=u(t)$, $v=v(t)$ to be a geodesic is $U\frac{\partial T}{\partial \dot v} - V\frac{\partial T}{\partial \dot u} = 0 \longrightarrow (1)$.

where $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) - \frac{\partial T}{\partial u}$; $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v} \rightarrow (2)$

given that the surface is with metric.

$v^2du^2 - 2uv\,du\,dv + 2u^2dv^2$, $u>0, v>0$.

$$ds^2 = v^2du^2 - 2uv\,du\,dv + 2u^2dv^2.$$

$$\dot s^2 = v^2\dot u^2 - 2uv\dot u\dot v + 2u^2\dot v^2.$$

$$\therefore T = \frac{1}{2}\dot s^2 = \frac{1}{2}(v^2\dot u^2 - 2uv\dot u\dot v + 2u^2\dot v^2).$$

$$\frac{\partial T}{\partial \dot u} = \frac{1}{2}(2v^2\dot u - 2uv\dot v) = v^2\dot u - uv\dot v.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) = v^2\ddot u + 2v\dot v\dot u - uv\ddot v - u\dot v^2 - \dot u v\dot v.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) = v^2\ddot u + v\dot u\dot v - uv\ddot v - u\dot v^2.$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}(-2v\dot u\dot v + 4u\dot v^2) = -v\dot u\dot v + 2u\dot v^2.$$

$$\frac{\partial T}{\partial \dot v} = \frac{1}{2}(-2uv\dot u + 4u^2\dot v) = -uv\dot u + 2u^2\dot v.$$

Problems :

$$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2u^2\dot{v}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = -uv\ddot{u} - \dot{u}^2 v - u\dot{v}\dot{u} + 4u\dot{u}\dot{v} + 2u^2\ddot{v}$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}(2v\dot{u}^2 - 2u\dot{u}\dot{v})$$

$$= v\dot{u}^2 - u\dot{u}\dot{v}$$

put $u = ct^3$ $\quad v = ct^2$ $\quad \therefore \frac{v^3}{u^2} = c.$

$\dot{u} = 3t^2c$ $\quad \dot{v} = 2ct$

$\ddot{u} = 6ct$ $\quad \ddot{v} = 2c.$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = v^2\ddot{u} + v\dot{u}\dot{v} - uv\ddot{v} - u\dot{v}^2$$

$$= (c^2t^4)(6ct) + (ct^2)(3ct^2)(2ct)$$
$$- (ct^3)(ct^2)(2c) - (ct^3)(4c^2t^2)$$

$$= 6c^3t^5 + 6c^3t^5 - 2c^3t^5 - 4c^3t^5$$

$$= 6c^3t^5.$$

$$\frac{\partial T}{\partial u} = -v\dot{u}\dot{v} + 2\dot{u}\dot{v}^2$$

$$= (-ct^2)(3t^2c)(2ct) + 2(3t^3c)(2ct)^2$$

$$= (-ct^2)(3t^2c)(2ct) + 2(6c\,3t^3c)(4c^2t^2)$$

$$= (c^2t^2)(3t^2c)(2ct) + 2(6c\,3t^3c)(4c^2t^2)$$

$$= 6c^{34}t^5 + 2(8b2\,t^5c^{34})$$

$$= 6c^4t^5 + 24t^4c^4$$

$$= -6c^3t^5 + 8c^3t^5$$

$$= 2c^3t^5.$$

$$U = 6c^3t^5 - 2c^3t^5$$
$$= 4c^3t^5.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = -uv\ddot{u} + 3\dot{u}\dot{v}u - \dot{u}^2v + 2u^2\ddot{v}.$$
$$= 3(3t^2c)(2ct)(ct^3)$$
$$= (-ct^3)(ct^2)(6ct) + 3(3t^2c)(2ct)(ct^3)$$
$$-(3t^2c)^2(ct^2) + 2(ct^3)^2(2c).$$
$$= -6c^3t^6 + 18c^3t^6 - 9t^6c^3 + 4c^3t^6.$$
$$= 13t^6c^3.\ 7c^3t^6.$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}(2v\dot{u}^2 - 2u\dot{u}\dot{v})$$
$$= v\dot{u}^2 - u\dot{u}\dot{v}$$
$$= (ct^2)(3t^2c)^2 - (ct^3)(3t^2c)(2ct).$$
$$= (ct^2)(9t^4c^2) - 6c^3t^6.$$
$$= 9c^3t^6 - 6c^3t^6$$
$$= 3c^3t^6.$$

1 → T
2 → t^2
3 → $\frac{\partial}{\partial}$
4 → game
summi
game
degree

$$V = 7c^3t^6 - 3c^3t^6$$
$$= 4c^3t^6.$$

$$U\frac{\partial T}{\partial \dot{u}} - V\frac{\partial T}{\partial \dot{u}}$$

$$U\frac{\partial T}{\partial v} - V\frac{\partial T}{\partial \dot{u}} = 0$$

$$(4c^3t^5)\cdot\frac{1}{2}(3c^3t^6) - 4c^3t^6(2c^3t^5)$$
$$= 12c^3\cdot c^3t^5(4\cdot 3t) - c^3t^5(4t\cdot 2)$$
$$= c^3t^5(12t) - c^3t^5(8t)$$
$$= c^3t^5(12t - 8t)$$
$$= c^3t^5(4t) =$$

$\therefore$ The given curve is a geodesics iff v satisfies a 2nd order D.E of the form.

$$(F^2 - EG)\ddot{v} + \left(GF_2 - \frac{GG_1}{2} - \frac{FG_2}{2}\right)\dot{v}^3 + \left(FF_2 - \frac{3}{2}FG_1 + GE_2 - \frac{EG_2}{2}\right)\dot{v}^2 + \left(\frac{3}{2}FE_2 + \frac{GE_1}{2} - EG_1 - FF_1\right)\dot{v} + \left(\frac{FE_1}{2} - EF_1 + \frac{EE_2}{2}\right)$$

$\therefore$ The given curve is a geodesics iff. v satisfies a 2^{nd} order D.E of the form.

$\ddot{v} + p\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0$ where P, Q, R & S are functions of u and v determined by E, F, and G.

canonical geodesic equations.

canonical geodesic equations are.

The geodesic equations

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\left(\frac{dT}{dt}\right) \frac{\partial T}{\partial \dot{u}},$$

$$V = \frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{v}},$$

where $T = \frac{1}{2}(E$ ... the geodesic equations are true for

Since the geodesic equations are true for any arbitary parameter t, it is true for the parameter is also.

If prime (,) denote the differentiation w.r.to s, the geodesic equations become,

$$U = \frac{d}{ds}\frac{\partial T}{\partial u'} - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{ds} \cdot \frac{\partial T}{\partial u'};$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{ds} \cdot \frac{\partial T}{\partial v'};$$

$$\longrightarrow (1).$$

where $T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$.

w.k.t.

$u' = \frac{du}{ds} = l$, $v' = \frac{dv}{ds} = m$ are the direction coefficients at a point on the curve and

$$El^2 + 2Flm + Gm^2 = 1 \longrightarrow (3)$$

$\therefore T = \frac{1}{2}$ (by (1) (2), (3)).

$\therefore \frac{dT}{ds} = 0.$

The geodesic equations becomes,

$$\left.\begin{aligned} U &= \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \\ V &= \frac{d}{ds}\frac{\partial T}{\partial v'} - \frac{\partial T}{\partial v} = 0 \end{aligned}\right\} \longrightarrow (4).$$

where $T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$.

Eqn (4) are called canonical equations for geodesics.

Remark:

$T = \frac{1}{2}$ only along the curve and not equal to $\frac{1}{2}$ identically $\forall$ u, v, u', v'.

Theorem:

For non-parametric curves a sufficient condition for a curve to be a geodesic is either $U = 0$ (or) $V = 0$.

proof:

w.k.t a necessary and sufficient condition for a curve to be a geodesic is

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} \text{ and } V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v}.$$

w.k.t

U and V satisfies the equation.

$$Uu' + Vv' = 0 \longrightarrow (1).$$

for non-parametric curves $u \neq$ constant and $v \neq$ constant and so $u' \neq 0$ and $v' \neq 0$.

$\therefore$ (1) implies that u and v are linearly dependent.

If $U = 0$, then clearly $V = 0$ and if $V = 0$, then clearly $U = 0$.

$\therefore$ a sufficient condition for a non-parametric curve to be a geodesic is either $U = 0$ (or) $V = 0$.

Normal property of geodesics.

The normal property which states that at every point on the geodesic its principal normal coincides with the surface normal and conversely every curve having this property is a geodesic. (or).

Any curve $u = u(t)$, $v = v(t)$ on a surface $r = r(u, v)$ is a geodesic if and only if the principal normal at every point on the curve as normal to the surface. (or).

* State and proof via normal property of geodesics.

proof:

We will establish that for a curve on a surface $r = r(u,v)$ to be a geodesic at a point p on the surface $r'' \vec{r_1} = 0$ and $r'' \vec{r_2} = 0$ showing that the principal normal $r'' = k\vec{n}$ of the curve is orthogonal to the tangential directions. direction r_1 and r_2 at P. so that the principal normal of the curve coincides with the surface car normal.

Since we use canonical geodesic equations in establishing the above we derive the canonical claim :-

$$\frac{\partial T}{\partial \dot{u}} = \dot{r} r_1 \quad \text{and} \quad \frac{\partial T}{\partial \dot{v}} = \dot{r} r_2 \longrightarrow (1).$$

$$u(t) = \dot{r} r_1 \;,\; v(t) = \ddot{r} r_2.$$

To prove the above identitles, let us. consider $\dot{r} = \frac{dr}{dt} = r_1 \dot{u} + r_2 \dot{v} \longrightarrow (2)$.

Hence $\dot{r}.\dot{r} = (r_1 \dot{u} + r_2 \dot{v})^2$

$$\dot{r}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2.$$

we can take, $T = \frac{1}{2}\dot{r}^2 \longrightarrow (3)$.

diff (3)

$$\frac{\partial T}{\partial u} = \dot{r}\frac{\partial \dot{r}}{\partial u} \;,\; \frac{\partial T}{\partial v} = \dot{r}\frac{\partial \dot{r}}{\partial v}.$$

$$\frac{\partial T}{\partial \dot{u}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{u}} \;,\; \frac{\partial T}{\partial \dot{v}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{v}}$$

using (2) and Diff partially.

$$\frac{\partial \dot{r}}{\partial \dot{u}} = r_1 \quad \text{and} \quad \frac{\partial \dot{r}}{\partial \dot{v}} = r_2.$$

$$\frac{\partial \dot{r}}{\partial u} = r_{11}\dot{u} + r_{21}\dot{v} \;,\; \frac{\partial \dot{r}}{\partial v} = r_{12}\dot{u} + r_{22}\dot{v} \longrightarrow (5)$$

where.

$$r_{11} = \frac{\partial^2 r}{\partial u^2} \qquad r_{12} = \frac{\partial^2 r}{\partial u \partial v}.$$

$$r_{21} = \frac{\partial^2 r}{\partial v \partial u} \qquad r_{22} = \frac{\partial^2 r}{\partial v^2}.$$

using (5) in (4) we get,

$$\frac{\partial T}{\partial v} = \dot{r}(r_{12}\dot{u} + r_{22}\dot{v}).$$

$$\frac{\partial T}{\partial \dot{u}} = \dot{r}r_1, \quad \frac{\partial T}{\partial \dot{v}} = \dot{r}r_2 \longrightarrow (6).$$

Now, let us find $U(t)$ and $V(t)$ as we've,

$$U(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}.$$

using (6).

$$U(t) = \frac{d}{dt}(\dot{r}r_1) - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v})$$

$$= \ddot{r}r_1 + \frac{dr_1}{dt}\dot{r} - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v}).$$

using $r_{12} = r_{21}$ we get $U(t) = \ddot{r}r_1$

$$= \ddot{r}r_1 + \dot{r}(r_{11}\dot{u} + r_{12}\dot{v}) - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v}) \longrightarrow (7)$$

further,

$$V(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}.$$

$$= \frac{d}{dt}(\dot{r}_1 r_2) - r(r_{12}\dot{u} + r_{22}\dot{v})$$

$$= \ddot{r}r_2 + \dot{r}\frac{d}{dt}(r_2) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v}).$$

$$= \ddot{r}r_2 + \dot{r}(r_{21}\dot{u} + r_{22}\dot{v}) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v}).$$

Hence $V(t) = \ddot{r}r_2$.

from (7) and (8).

we get the required identities (1),

Instead of taking the parameters t, we can use as the parameters. In the case. we have to replace t by s in the above working so that we have.

$U(s) = \dots''$ and $V(s) = r'' \cdot r_2$.

Hence the canonical geodesic eqn $v(s)=0$,

$v(s)=0$.

$\Rightarrow r'' r_1 = 0$ and $r'' \cdot r_2 = 0$.

(1) $\Rightarrow$. r'' is $\perp$ to the vectors r_1 and r_2.

lying in the tangent plane at p.

Hence r'' is the surface normal at p.

But,

$$r'' = \frac{d^2 r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{dt}{ds} = \kappa n.$$

so that r'' is along the principal normal of the geodesic at p. hence the principal normal at every point of the geodesic is normal. to the surface at p. since all the above steps are eversible converse is also true.

Q Problem:

prove that every helix on a cylinder is a geodesic.

sol:-

Let γ be a helix on the cylinder

Then $\vec{a} \cdot \vec{t} =$ constant.

Diff w.r.to s.

$$\vec{a}' \cdot \vec{t} + \vec{a} \cdot \vec{t}' = 0.$$

$$\vec{a} \cdot \vec{t}' = 0$$

$$\vec{a} \cdot k\vec{n} = 0$$

$$\vec{a} \cdot \vec{n} = 0$$

Also, $\vec{t} \cdot \vec{n} = 0$.

$\vec{n}$ is parallel to $\vec{a} \times \vec{t}$.

Hence the vector $\vec{a}$ and $\vec{t}$ are tangential to the surface of the cylinder at p.

Hence the unit surface normal $\vec{N}$ is parallel to the vector $\vec{a} \times \vec{t}$

i.e., $\vec{n}$ is parallel to the $\vec{N}$.

Thus γ is geodesics on the surface.

Hence every helix on a cylinder is a geodesic

Christofell symbols.

*. Let $\vec{r}' = \vec{r_1} u' + \vec{r_2} v'$

Diff $\vec{r}'' = \vec{r_1} u'' + u'(\vec{r_{11}} u' + \vec{r_{12}} v') + \vec{r_2} v'' + v'(\vec{r_{21}} u' + \vec{r_{22}} v')$.

$\vec{r}'' = \vec{r_1} u'' + \vec{r_2} v'' + \vec{r_{11}} u'^2 + \vec{r_{22}} v'^2 + 2\vec{r_{12}} u'v'$.

The geodesic eqns are,

$\vec{r}'' \cdot \vec{r_1} = 0$.

$\vec{r}'' \cdot \vec{r_2} = 0$.

(1) $\Rightarrow (\vec{r_1} u'' + \vec{r_2} v'' + \vec{r_{11}} u'^2 + 2\vec{r_{12}} u'^2 + \vec{r_{22}} v'^2)\vec{r_1} = 0$

$\vec{r_1}^2 u'' + \vec{r_1} \cdot \vec{r_2} v'' + \vec{r_{11}} \vec{r_1} u'^2 + 2\vec{r_{12}} \vec{r_1} \cdot u'v' + \vec{r_{22}} \vec{r_1} v_2'^2 = 0$.

$Eu'' + Fv' + \Gamma_{111} u'^2 + 2\Gamma_{112} u'v' + \Gamma_{122} v'^2 = 0$.

where formula $\Gamma_{ijk} = \vec{r_i} \cdot \vec{r_{jk}}$ ($i,j,k = 1,2,3$).

$|||^{y}$ becomes.

$Fu'' + Gv'' + \Gamma_{211} u'^2 + 2\Gamma_{212} u'v' + \Gamma_{222} v'^2 = 0$.

The coefficients Γ_{ijk} are called christofell symbols of the first kind.

Also, $\Gamma^1_{jk} = H^{-2}[G\Gamma_{1jk} - F\Gamma_{2jk}]$.

$\Gamma^2_{jk} = H^{-2}(E\Gamma_{2jk} - F\Gamma_{jk}]$

The coefficients Γ^i_{jk} are called the christofell symbols of the second kind.

The resulting eqns are,

$u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2 = 0$.

$v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2 = 0$.

Geodesic parallel:

Theorem:-

suppose a family of geodesics is given and that a parameter system is chosen. so that geodesics of the family are curves $v=$ constant and their orthogonal trajectories are the curves $u=$ constant.

Then $F=0$ and w.k.t,

$EE_2 + FE_1 - 2EF_1 = 0$.

For the curves $v=$ constant to be geodesics becomes $E_2=0$.

$\therefore$ The metric is of the form.

$ds^2 = E(u)du^2 + G(u,v)dv^2$.

consider the distance between any two of the orthogonal trajectories along the geodesics $v=c$,

Along $v=c$, $dv=0$ & $ds=\sqrt{E}\,du$ so, that the distance is $\int_{u_1}^{u_2} \sqrt{E(u)}\,du$,

a number of dependent of c.

Thus the distance is the same along whichever geodesic $v=$ constant

It is measured. Because of this the orthogonal trajectories are called geodesic parellel.

Defn:-

The orthogonal trajectories of the geodesic u = constant of the family of curves v = constant are called "geodesic parellel".

Problem 1:-

Find the geodesic on a surface of revolution.

Soln

The surface of revolution is

$$\vec{r} = (g(u)\cos v,\ g(u)\sin v,\ f(u))$$

$$E = \vec{r_1}^2 = g_1^2 + f_1^2.$$

$$F = \vec{r_1}\cdot\vec{r_2} = 0.$$

$$G = \vec{r_2}^2 = g^2.$$

The metric is

$$ds^2 = (g_1^2 + f_1^2)\,du^2 + g^2 dv^2.$$

then,

$$T = \frac{1}{2}\left[(f_1^2 + g_1^2)\,u'^2 + g^2 v'^2\right] \longrightarrow (1).$$

where $f_1 = \dfrac{df}{du}$.

$$g_1 = \frac{dg}{du}.$$

The canonical geodesic eqn are,

$$u = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \longrightarrow (2)$$

$$v = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \longrightarrow (3).$$

Diff (1) w.r.t to v.

$$\frac{\partial T}{\partial v} = 0.$$

(2) becomes,

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) = 0$$

$$\cdot \frac{d}{ds}(g^2 v') = 0.$$

$g^2 v' = \alpha$ where α is a constant.

Hence the geodesic on a surface of revolution is $g^2 v' = \alpha \longrightarrow (4)$

If $\alpha = 0$, $g^2 v' = 0$.

$v =$ constant.

$\therefore$ Every meridian is a geodesic.

If α is positive,

(4) can be written as,

$$g^4 v' = \alpha.$$

$$g^4 dv^2 = \alpha^2 ds^2$$

$$= \alpha^2 [(f_1^2 + g_1^2) du^2 + g^2 dv^2)].$$

$$= \alpha^2 [(f_1^2+g_1^2) du^2.$$

$$g^4 dv^2 = \alpha^2 g^2 dv^2 = \alpha^2 (f_1^2 + g_1^2) du^2$$

$$g^4 dv^2 = \alpha^2 g^2 dv^2 = \alpha^2 (f_1^2 + g_1^2) du^2 \longrightarrow (5)$$

$$g^2 (g^2 - \alpha^2) dv^2 = \alpha^2 (f_1^2 + g_1^2) du^2$$

$$dv = \pm \frac{\alpha}{g} \frac{\sqrt{f_1^2+g_1^2}}{\sqrt{g^2-\alpha^2}}\, du.$$

$$v = \beta \pm \alpha \int \left(\frac{f_1^2 + g_1^2}{g^2 - \alpha^2} \right)^{1/2} \frac{du}{g}, \text{ if } g^2 - \alpha^2 \neq 0.$$

This is of the form $v = \alpha\, \phi(u, \alpha) + \beta$.

where α, β are constants,

Hence the geodesic on a surface of relation is revolution is $v = \beta \pm \alpha \int \left(\frac{f_1^2 + g_1^2}{g^2 - \alpha^2} \right) \frac{du}{g}$.

if $g^2 \neq \alpha^2$. If $g^2 = \alpha^2$.

(5) becomes,

$$\alpha^2 (f_1^2 + g_1^2)\, du^2 = 0$$

$$\Rightarrow du^2 = 0$$

$$\Rightarrow du = 0$$

$$\Rightarrow u = \text{constant}.$$

Also $F = 0$,

Hence the curve $u = c$ (parallel) is a geodesic iff G is a fn of v only,

ie) $G_1 = 0$.

But, $G = g^2$

$G_1 = 0$ iff g is a constant.

ie) $G_1 = 0$ iff u is a constant.

$\therefore$ The curve is a geodesic iff

$g_1(c) = 0$.

Since g is the radius of the parallel.

$u = c$ on the surface of revolution.

A parallel geodesic if its radius is stationary.

Theorem:-

#. prove that geodesics on a right circular, cylinder are helices.

soln

we know that,

If a geodesic cuts the meridian at any point at an angle ϕ.

$u \sin\phi = h$ (constant).

where, u is the distance of the point from the axis,

Here the surfaces of revolution is a right circular cylinder.

w.k.t,

The meridian of surface of revolution are the generation of the right cylinder.

The distance of every point on the generator from the axis is constant.

$u = a$ (constant).

$\sin\phi = \frac{h}{a}$ = constant

Hence a geodesic ~~on~~ a cylinder cuts all the generator at a constant angle.

Thus the geodesic is a helix.

Show that the edge of the regression of the oscillating developable is the given curve.

To study the oscillating developable.

Let $\vec{r} = \vec{r}(s)$.

The given space curve and the eqn of the oscillating plane at a point on the curve is.

$(\vec{R} - \vec{r})\vec{b} = 0.$

$\vec{R}$ is the position vector of the current point in the oscillating plane.

Let $F(s) = (\vec{R} - \vec{r})\vec{b} = 0.$

$f'(s) = 0$

i.e) $(\vec{R} - \vec{r})(-\tau\vec{n}) + (0 - \vec{t})\vec{b} = 0.$

$(\vec{R} - \vec{r})\vec{n} = 0 \longrightarrow ②.$

which is the eqn ① & ② is the tangent line to the curve at point P.

The oscillating developable is the surface of generator by the tangent line.

Edge of regression,

Diff w.r to s to eqn ②.

$(\vec{R} - \vec{r})(0 - k\vec{t}) + (0 - \vec{t})\vec{R} = 0$

$(\vec{R} - \vec{r})\vec{t} = 0 \longrightarrow ③.$

from ① & ② & ③.

$(R - r) = 0$

$\vec{R} = \vec{r}$.

The edge of regression consside with the given curve is itself.

Polar developable:..

The family of normal planes to a skew curve form the polar developable of the given curve.

The developable of the normal planes twice curve is called polar developable and the generated called the polar lines.

Rectifying developable:.

The envelopable of the rectifying plane of the curve is called the rectifying developable. and its generated are the rectifying lines this the rectifying line at a p of the curve is the intersection of the consequentive. lines of rectifying planes.

Geodesic curvature:. (K_g)

The geodesic curvature K_g of any curve is defined as the magnitude of the geodesic curvature. vector with a sign attached, +ve or -ve according as the angle between the tangent and the geodesic curvature vector is $\frac{1}{2}z$ or $-\frac{1}{2}z$.

Prove that the edge of regression of the rectifying developable.

Soln

Let $\vec{r} = \vec{r}(s)$ be the given space curve and the eqns of the rectifying plane and a point on the curve is given by $(R-\vec{r}).\vec{n} = 0$ $\rightarrow$ ①

where $\vec{R}$ & $\vec{r}$ is the function of the s.

Diff w.r.to s.

$(\vec{R}-\vec{r})(\tau\vec{b} - k\vec{t}) + (0-\vec{t})\vec{n} = 0.$

$(\vec{R}-\vec{r})(\tau\vec{b} - k\vec{t}) = 0$ $\longrightarrow$ ②.

The intersection of eqn ① & ② is the line passing through the point p on the R curve and is $\perp^r$ to $\vec{n}$.

hence it is parallel to $\vec{n} \times (\tau\vec{b} - k\vec{t})$.

ie) $\tau\vec{t} + k\vec{b}$. $R-\vec{r} = \lambda(\vec{n})$ $n\times\tau b - k\, n\times t$ $\tau t + kb$.

Let ϕ be the angle taking by the vector $\tau\vec{t} + k\vec{b}$ with this vector $(\vec{t})$.

$$\cos\psi = \frac{(\tau\vec{t} + k\vec{b})\cdot\vec{t}}{\sqrt{\tau^2+k^2}} = \frac{\tau}{\sqrt{\tau^2+k^2}}.$$

$\tau(-k)$ $-\tau(k)$

$b = t\times n$ $= -kt + \tau b$ $R-\vec{r} = \lambda$

$$\sin\varphi = \frac{(\tau\vec{t}+k\vec{b})\vec{b}}{\sqrt{\tau^2+k^2}} = \frac{k}{\sqrt{\tau^2+k^2}}$$

$$\tan\varphi = \frac{\sin\phi}{\cos\phi} = \frac{k}{\tau}$$

also, $\vec{R}-\vec{r} = \lambda(\tau\vec{t}+k\vec{b}) \longrightarrow (A)$

the line is inclined to the tangent at an angle φ such that $\tan\phi = \frac{k}{\tau}$.

To find the edge of regression diff eqn (2)

we get,

$$[(\vec{R}-\vec{r})(\tau'\vec{b}+\tau(-\tau\vec{n})) - (k'\vec{t}+k(k\vec{n}))] + [0-\vec{t}](\tau\vec{b}-k\vec{t}) = 0.$$

$$(\vec{R}-\vec{r})[\tau'\vec{b}-\tau^2\vec{n}-k'\vec{t}-k^2\vec{n}+k'] = 0.$$

$$(R-\vec{r})[\tau'\vec{b}-k'\vec{t}] - (\tau^2+k^2)\vec{n}(R-\vec{r})+k = 0.$$

$$(R-\vec{r})(\tau'\vec{b}-k'\vec{t})+k = 0 \longrightarrow (3).$$

we have already denote that $(\vec{R}-\vec{r})$ is parallel to $\tau\vec{t}+k\vec{b}$.

$$(\vec{R}-\vec{r}) = \lambda(\tau\vec{t}+k\vec{b})$$

where λ is to be determined.

we put this values in (3), we get,

$$\lambda(\tau\vec{t}+k\vec{b})\cdot\{(\tau'\vec{b}-k'\vec{t})+k\} = 0.$$

$$\lambda\tau'k - \tau\lambda k'$$

$\lambda k\tau' - \lambda \tau k' + k = 0.$ $\lambda\{\tau t + kb\}\{\tau' b - k' t\}$

$(\lambda\tau t + \lambda kb)(\tau' b - k' t)$

$\lambda(\tau k' - k\tau') = k.$ $\lambda\tau\tau' tb$ $\lambda kk' bt$

$$\lambda = \frac{k}{\tau k' - k\tau'}$$ put get in (A).

$t = n \times b$
$n = b \times t$
$b = t \times n$

$$(\vec{R} - \vec{r}) = \frac{k}{\tau k' - k\tau'}(\tau\vec{t} + k\vec{b}).$$

$$\vec{R} = \vec{r} + \frac{k(\tau\vec{t} + k\vec{b})}{\tau k' - k\tau'}$$ //

$\vec{R} = \vec{r} + u\vec{l}(t)$

developable associated with curve on a surface.

To prove the theorem.

[we consider the surface formed by a normals along the curve $\vec{r} = \vec{r}(s)$ any point on the surface will have position vector $\vec{R} = \vec{r}(s) + v\vec{N}(s)$.

here s and v are parameters.

simply we have write.

$\vec{R} = \vec{r} + v\vec{N}$

$\vec{R_1} = \vec{t} + \vec{v}\vec{N}'$

$\vec{R_2} = \vec{N}$

$\vec{R_{21}} = \vec{N}' = \vec{R_{12}}$

$\vec{R_{22}} = 0$

But $[\vec{R_1}, \vec{R_2}, \vec{R_{12}}] = HM$ and

$[\vec{R_1}, \vec{R_2}, \vec{R_{22}}] = HN,$

$$HM = [\vec{E} + V\vec{N}, \vec{N}, \vec{N}']$$

$$= [\vec{E}, \vec{N}, \vec{N}'] + [V\vec{N}, \vec{N}, \vec{N}']$$

$$= [\vec{E}, \vec{N}, \vec{N}'].$$

$$H\vec{N} = [\vec{E} + V\vec{N}, \vec{N}, 0] = 0.$$

$$\Rightarrow N = 0 \; //.$$

Gauss Bonnet theorem:

For a geodesic triangle ABC, formed by geodesics arcs AB, BC, CA and enclosing a simply connected region R. the access is.

$2\pi - (\pi - A) - (\pi - B) - (\pi - C) = A + B + C - \pi.$

where ABC are the interior angles of. the triangle. thus the excess is the. excess of A+B+C over its euclidean value π, a fact which accounts. historically for our use of the word excess. the total curvature of a geodesic triangle ABC is therefore equal to

$A + B + C - \pi.$

more generally for a geodesic polygon. of any number of sides (geodesic arcs). the total curvature is equal to 2π. minus the sum of the exterior angle. the excess of the interior angle. i.e) the

Theorem:-

If $r=r(s)$ is the position vector of a point P of a curve on a surface then

(i) $k_g = [N, r', r'']$

(ii) $k_g = s^{-3}[N, \dot{r}, \ddot{r}]$.

Proof:

The geodesic curvature vector is orthogonal to the unit tangent vector. $r' = \frac{dr}{ds}$ at P.

Since the geodesic curvature vector $\lambda r_1 + \mu r_2$ lies in the tangent plane at P. it is orthogonal to this surface normal N at P. Thus the geodesic curvature vectors is orthogonal to both N and r' and therefore it is parallel to the unit vector $N \times r'$. since k_g is the magnitude of the geodesic curvature vector. we can take the geodesic curvature vector $\lambda r_1 + \mu r_2 = k_g (N \times r')$ $\longrightarrow$ (1).

We know that

$r'' = k_n N + \lambda r_1 + \mu r_2$ $\longrightarrow$ (2).

Using (1) in (2).

then $r'' = k_n N + \lambda r_1 + \mu r_2$.

$= k_n N + k_g (N \times r')$ $\longrightarrow$ (3).

on both sides of ③.

$(N\times r')\cdot r'' = [k_n N + k_g (N\times r')]\cdot(N\times r')$. ④

$N(N\times r') = 0$ and $(N\times r')(N\times r') = 1$.

we get from ④, $k_g = [N, r', r'']$.

(ii) we shall rewrite the formula now,

i) by using any parameters t.

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{ds}\cdot\frac{ds}{dt} = r'\dot{s}$$

$$\ddot{r} = \frac{d}{dt}(r'\dot{s}) = \frac{d}{ds}(r'\dot{s})\frac{ds}{dt} = r''\dot{s}^2 + r'\ddot{s}$$

since

$$r'\dot{r}\times r'\ddot{s} = 0.$$

$$\dot{r}\times\ddot{r} = r'\dot{s}\times(r''\dot{s}^2 + r'\ddot{s}) = r'\times r''\dot{s}^3.$$

so that,

$$r'\times r'' = \frac{1}{\dot{s}^3}(\dot{r}\times\ddot{r}).$$

Hence from the formula (i) we have,

$$k_g = N\cdot(r'\times r'') = \frac{1}{\dot{s}^3}[N, \dot{r}, \ddot{r}].$$

$v=0$ $u=$

2. Theorem

If k_a and k_b denote the geodesic curvature of the parameter curves $v=$constant and $u=$constant respectively.

(i) $k_a = \frac{1}{2H} E^{-3/2}[2EF - EE_2 - FE_1]$

Proof:

The parametric curve v=constant.

let us take u itself as the parametric,

so that $\dot{u}=1$ and $\dot{v}=0$.

Now,

$$T=\frac{1}{2}[E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2]$$

$T=\frac{1}{2}[E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2]$

Hence,

$$\frac{\partial T}{\partial \dot{u}}=E\dot{u}+F\dot{v}\,;\quad \frac{\partial T}{\partial \dot{v}}=F\dot{u}+G\dot{v}.$$

$$\frac{\partial T}{\partial u}=\frac{1}{2}[E_1\dot{u}^2+2F_1\dot{u}\dot{v}+G_1\dot{v}^2].$$

$$\frac{\partial T}{\partial v}=\frac{1}{2}[E_2\dot{u}^2+2F_2\dot{u}\dot{v}+G_2\dot{v}^2]$$

From $\dot{u}=1$, $\dot{v}=0$.

$$\frac{\partial T}{\partial \dot{u}}=E,\quad \frac{\partial T}{\partial \dot{v}}=F.$$

$$\frac{\partial T}{\partial u}=\frac{1}{2}E_1,\quad \frac{\partial T}{\partial v}=\frac{1}{2}E_2 \longrightarrow (1)$$

Further,

$$U=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)-\frac{\partial T}{\partial u}$$

$$=\frac{\partial E}{\partial u}\dot{u}+\frac{\partial E}{\partial v}\dot{v}-\frac{\partial T}{\partial u}$$

$$V=\frac{d}{dt}\frac{\partial T}{\partial \dot{v}}-\frac{\partial T}{\partial v}\longrightarrow (2)$$

using (1) in (2),

$$U=\frac{d}{dt}(E)-\frac{1}{2}E_1=E_1\dot{u}+E_2\dot{v}-\frac{1}{2}E_1$$

$$=\frac{1}{2}E_1\longrightarrow (3)$$

$$v = \frac{d}{dt}(F) - \frac{1}{2}E_2 = F_1\dot{u} + F_2\dot{v} - \frac{1}{2}E_2$$

$$= F_1 - \frac{1}{2}E_2. \longrightarrow (4).$$

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$, becomes

$$\dot{s}^2 = E, \text{ so that } \dot{s} = \sqrt{E} \longrightarrow (5)$$

$$\boxed{k_g = \frac{1}{H\dot{s}^3}\left[\frac{\partial T}{\partial \dot{u}} v(t) - \frac{\partial T}{\partial \dot{v}} u(t)\right]} \longrightarrow (6).$$

$$= \frac{1}{H \cdot E^{3/2}}\left[E\left(F_1 - \frac{1}{2}E_2\right) - F \times \frac{1}{2}E_1\right].$$

$$= \frac{1}{H} E^{-3/2}\left[EF_1 - \frac{1}{2}EE_2 - F\frac{E_1}{2}\right]$$

$$k_a = \frac{1}{2H} E^{-3/2}\left[2EF_1 - E_2E - FE_1\right].$$

(ii) Let $k_b = \frac{1}{2H} G^{-3/2}\left[FG_2 + GG_1 - 2G_1\right]$

Sol:

Let us treat v itself as the parameter.

so that $\dot{v} = 1$ and $\dot{u} = 0$.

$$\frac{\partial T}{\partial \dot{u}} = F_1, \quad \frac{\partial T}{\partial \dot{v}} = G, \quad \frac{\partial T}{\partial u} = G_1, \quad \frac{\partial T}{\partial v} = F \longrightarrow (7)$$

$$U = \frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial v}\dot{v} - \frac{1}{2}G_1 = F_2 - \frac{1}{2}G_1 \longrightarrow (8)$$

$$V = \frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial v}\dot{v} - \frac{1}{2}G_2 = G_2 - \frac{1}{2}G_2$$

$$= \frac{1}{2}G_2 \longrightarrow (9)$$

since $\dot{v} = 1$ and $\dot{u} = 0$ $\dot{s} = \sqrt{G} \longrightarrow (10)$

using (7), (8), (9) and (10) in (6).

$$\frac{1}{\quad} G^{-3/2}\left[FG_2 + GG_1 - 2GF_2\right].$$

Gauss - Bonnet.

For any curve c which encloses a simply connected region R on surface on c is equal to the total curvature of R.

$c = t_c$ of R.

Proof:

We shall use Liouville's formula for kg and find $\int kg\, ds$ with the help of Green's in the plane for a simply connected region R bounded by c. so we shall quote this Green's theorem.

Green's theorem.

If R is a simply connected region bounded by a closed curve c, then,

$$\int_c P\,du + Q\,dv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv.$$

where P and Q are differentiable fn. of u and v in R.

Proof!

from the Liouville's formula.

$kg = Q' + Pu' + Qv'$.

Integrating along the curve c we've,

$$\int_c kg\, ds = \int_c (Q' + Pu' + Qv').\, ds.$$

$$= \int_c (dQ + P\,du + Q\,dv) \longrightarrow (1).$$

where θ is the angle ...

curve c and the parametric curve,

v = constant and. p and Q are differentiable

function of u, v.

Let us suppose the simple closed.

curve contains a finite number of

arcs starting from A. then at each

point Q the arc there passes A curve.

v = constant making an angle θ with a.c.

Hence we describe the curve c. the

tangent at various members of the

family v = constant described in the

positive sense returns to the starting

point after increasing the angle of

rotation by 2π.

This increase 2π after complete rotation.

in the positive sense also includes.

the angle between the tangent at.

the finite number of vertices.

$\int_c d\theta + \sum_{r=1}^{n} d r = 2\pi$

$$\int_c d\theta + \sum_{r=1}^{n} d r = 2\pi \longrightarrow (2).$$

$$\int_c d\theta + \sum_{r=1}^{n} d r = 2\pi$$

from the def ex $c = 2\pi - \sum_{r=1}^{n} d r - \int_c Kg \, ds.$

using (1) and (2) in (3)

$$ex\ c = 2\pi - \left[2\pi - \int_c d\theta\right] - \left[\int_c d\theta + p\,du + Q\,dv\right]$$

Thus $ex\ c = -\int_C (p\,du + Q\,dv) \longrightarrow (4)$

since R is a simply connected region p and Q are differentiable functions of. u,v, we have by Green's theorem.

$$\int_C (p\,du + Q\,dv) = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial p}{\partial v}\right) du\,dv \longrightarrow (5)$$

since the surface elements $ds = H\,du\,dv$.

we write (5) as,

$$\int_C (p\,du + Q\,dv) = \frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) ds \longrightarrow (6).$$

using (5) in (4) we get,

$$ex\ c = -\frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) ds \longrightarrow (7).$$

If we take,

$$K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$$

$$ex\ c = \iint_R K\,ds \longrightarrow (8).$$

when K is a function u and v and it is. independent of the c and defined over the region R of the surface.

next we shall show that the ex c is uniquely determined by K. If K is not unique let $\bar{K}$ be such that.

$$\iint_R \bar{K}\,ds = ex\ c \longrightarrow (9)$$

using ⑧ and ⑨

$$\iint_R (\bar{K}-K)\,ds = 0 \longrightarrow ⑩$$

for every region R, now let $\bar{K} \neq K$ at some point at A. then we must have $\bar{K} > K$ or $\bar{K} < K$ at A.

let us first consider $\bar{K} > K$. since the given surface is of class 3. $\frac{\partial Q}{\partial u}$ and $\frac{\partial P}{\partial v}$ are continuous R.

so that ∃ a small region R_1 of R containing the point A such that $\bar{K}-K>0$ at every p of R_1 for this region R contain R_1. $\iint_{R_1} (\bar{K}-K)\,ds > 0$.

which contradicts ⑩.

we get a similar contradiction.

contradiction $\iint_{R_1} (\bar{K}-K)\,ds > 0$ at A where $\bar{K} < K$ then

prove that $\bar{K} = K$ at every point of R.

That it is K is uniquely determined as a funs of u and v.

defining $\int_R K\,ds$ the total curvature of R we have proved that the total curvature is exactly the ex c in any region R enclosed by C//.

↓

continuation back side.

Gaussian curvature

Defn:-

If ka and kb are the principal curvature the Gaussian curvature, denoted by K is defined by

$K = ka \cdot kb$.

from the product of the roots of the roots of the eqn ① we have

$K = ka kb$.

$[\because k^2(EG-F^2) - k(EN+GL-2FM) + (LN-M^2)]$

$K = \frac{LN-M^2}{EG-F^2}$ $[\because ab = c/a]$.

Mean curvature.

If ka and kb are the principal curvature at a point on a surface, then the mean curvature denoted by μ is defined by.

$\mu = \frac{1}{2}(ka+kb)$.

Sum of the roots of the equation ①

we have,

$$\mu = \frac{EN+GL-2FM}{2(EG-F^2)} \quad (\because a+b = -b/a).$$

Note:1.

we shall prove that latter the definition of Gaussian curvature denoted by existence above def are equivalent.

The value of the P.C is independent of the parametric system chosen.

Let us consider the parametric transformation

$u = \phi(u', v')$ $\quad v = \psi(u', v')$.

w.k.t, $L'N' - M'^2 = J^2(LN - M^2)$

where J is Jacobian.

$E'G' - F'^2 = J^2(EG - F^2)$

$$K' = \frac{LN - M^2}{EG - F^2}$$

$$K' = K.$$

showing that the G.C is independent. Parametric system chosen.

Note: 3.

using the defn of G.C we can, characteristic differentiable points on the surface as follows from the eqn,

$$K = \frac{LN - M^2}{EG - F^2}, \quad EG - F^2 \neq 0$$

$H^2 = EG - F^2$ is always +ve.

case (i).

If K is +ve at a point P on the surface then $LN - M^2 > 0$ which means that is an eliptic point.

Hence a point on a surface is eliptic point. iff two principal curvature at a point are of same sign.

case (ii).

If K is −ve at a point p on a surface then $LN-M^2 < 0$ which means that an hyperbolic point.

Hence a point on a surface is a hyperbolic point, iff two principal direction at a point are of opposite signs.

case (iii).

If $K=0$ at a point p on the surface then $LN-M^2=0$ which means that is an Parabolic point.

hence a point on surface is a parabolic atleast one of principal curvature is zero.

A curve on a surf whose at each point is along

Theorem:

The principal direction are given by,

$$(EM-FL)l^2 + (EN-GL)lm + (FN-GM)m^2 = 0.$$

proof:

w.k.t, the principal curvature is given by,

$$Ll + Mm - \lambda El - \lambda Fm = 0 \longrightarrow (1).$$

$$Ml + Nm - \lambda Fl - \lambda Gm = 0 \longrightarrow (2).$$

$$Ll + Mm = \lambda(El + Fm).$$

$$Ml + Nm = \lambda(Fl + Gm).$$

$$\frac{\lambda}{1} = \frac{Ml+Nm}{Ll+Mm} = \frac{x(Fl+Gm)}{x(El+Fm)}$$

$(El+Fm)(Ml+Nm) = (Ll+Mm)(Fl+Gm)$.

$(El+Fm)(Ml+Nm) - (Ll+Mm)(Fl+Gm) = 0$

$(El+Fm)(Ml+Nm) - (Ll+Mm)(Fl+Gm)$

$(EMl^2 + ENlm + FMlm + FNm^2) -$

$(LFl^2 + GLlm + FMml + GMm^2) = 0$.

$(EM-FL)l^2 + (FN-GM)m^2 + (EN-GL)lm = 0 \longrightarrow (3)$

which is quadratic in l.

The above defn gives to principal curvature on the surface we shall known that the roots of the above defn are real and distinct.

The discriminant of quadratic (3) is

$(EN-GL)^2 - 4(EM-FL)(FN-GM) \longrightarrow (4)$

$[\because al^2 + blm + cm^2 = b^2 - 4ac]$.

$c = a+b$

we shall rewrite by completing the squares for this, let us consider,

$$\overset{c}{(FN-GM)} = F/E\,\overset{a}{(EN-GL)} + G/E\,\overset{b}{(EM-FL)} \longrightarrow (5)$$

sub (5) in (4), we get,

$\Rightarrow (EN-GL)^2 - 4(EM-FL)\,[F/E\,(EN-GL) + G/E\,(EM-FL)]$.

$\Rightarrow (EN-GL)^2 - \frac{4F}{E}(EM-FL)(EN-GL) - 4G/E\,(EM-FL)^2 + \boxed{\frac{4F^2}{E^2}(EM-FL)^2} - \frac{4F^2}{E^2}(EM-FL)^2$

$$\Rightarrow [(EN-GL) - \frac{2F}{E}(EM-FL)]^2 + \frac{4(EG-F^2)}{E^2}(EM-FL)^2.$$

since $EG-F^2>0$ the discriminant of ③ always +ve.

so that the roots of eqn ③ are real and distinct.

If we take $l = \frac{du}{ds}$, $m = \frac{dv}{ds}$ then from ③

direction are given by,

$$(EM-FL)\frac{du^2}{ds^2} + (EN-GL)\frac{dudv}{ds^2} + (FN-GM)\frac{dv^2}{ds^2} = 0$$

$$(EM-FL)du^2 + (EN-GL)dudv + (FN-GM)dv^2 = 0.$$

Identifying above eqn, we have

$$Pdu^2 + Qdudv + Rdv^2 = 0.$$

condition are orthogonality.

$$(FN-GM) - F(EN-GL) + G(EM-FL) = 0.$$

$\therefore$ principal direction at every point on the surface are orthogonal.

Umbilic

Defn:

A point on a surface is called umbilic if at that point $L/E = M/F = N/G$ is true.

Note:1.

$$L/E = M/F = N/G = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}.$$

which represent a normal curvature hence is an umbilic. The normal curvature is same in all direction.

NOTE 2!

Eqn ① in theorem ① giving a principal direction is also the same as $(FN-GM)dv^2 + (EN-GL)dudv + (EM-FL)du^2 = 0$. which can be written in determinant form.

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

Problem!.

find the principal direction and principal curvature at a pt on the surface is $x = a(u+v)$, $y = b(u-v)$ $z = uv$.

Sol

The position vector also at any point on the surface.

$r = (a(u+v), b(u-v), uv)$.

$r_1 = (a, b, v)$, $r_2 = (a, -b, u)$.

$$r_1 \times r_2 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ a & b & v \\ a & -b & u \end{vmatrix}$$

$$= \vec{t}\,(bu+bv) - \vec{n}\,(au - av) + \vec{b}\,(-ab-ab)$$

$$= \vec{t}\,(u+v)b - \vec{n}\,(u-v)a + \vec{b}\,(-2ab)$$

$$= \vec{t}\,(u+v)b + \vec{n}\,(v-u)\vec{a} + \vec{b}\,(-2ab).$$

$(r_1 \times r_2) = (b(u+v), a(v-u), -2ab)$.

$r_1 = a, b, v$ $\quad r_2 = a, -b, u$

$r_{11} = (0,0,0)$ $\quad r_{22} = (0,0,0)$.

$r_{12} = (0,0,1)$ $\quad r_{21} = (0,0,1)$.

$r_1 = (a, b, v)$ $r_2 = (a, -b, u)$

$E = r_1^2 = a^2+b^2+v^2$ $\quad = (a^2, -b^2, uv)$.

$F = r_1 \cdot r_2 = a^2 - b^2 + uv$

$G = r_2^2 = a^2+b^2+u^2$.

$L = r_{11} \cdot N = 0$.

$N = r_{22} \cdot N = 0$

$$M = \frac{r_{12} \cdot N}{H} = \frac{-2ab}{H}$$

$$LN - M^2 = -\left(-\frac{2ab}{H}\right)^2 = \frac{-4a^2b^2}{H^2}.$$

which always -ve, Hence every point on a surface is a hyperpolic point.

The principal direction is given by,

$$\begin{vmatrix} dv^2 & -du dv & du^2 \\ E & F & G \\ 0 & M & 0 \end{vmatrix} = 0.$$

$dv^2(-GM) + du^2(EM) = 0$

$\therefore M \neq 0$.

sub. the value of E & G,

$(a^2+b^2+v^2)\, du^2 - (a^2+b^2+u^2)\, dv^2 = 0$.

$(a^2+b^2+v^2)\, du^2 = (a^2+b^2+u^2)\, dv^2$.

$$\frac{du}{\sqrt{a^2+b^2+u^2}} = \frac{dv}{\sqrt{a^2+b^2+v^2}}$$ giving the.

Principal direction.

principal curvature given by the eqn.

$(EG-F^2)k^2 - (GL - EN - 2FM)k + (LN - M^2) = 0.$

$\therefore (EG-F^2) = H^2$, $G = a^2+b^2+u^2$ & $L = 0$.

$E = a^2+b^2+v^2$ & $N = 0$, $F = a^2-b^2+uv$.

$$LN - M^2 = \frac{-4a^2b^2}{H^2} \quad \& \quad M = \frac{-2ab}{H}.$$

$$H^2k^2 + 2(a^2-2b^2+uv)\left(\frac{-2ab}{H}\right)k - \frac{4a^2b^2}{H^2} = 0.$$

$-4a$

$$H^2k^2 + \left(\frac{-4a^2b}{H} + \frac{4ab^3}{H} - \frac{4abuv}{H}\right)k - \frac{4a^2b^2}{H^2} = 0$$

$H^2k^2 - (4a^2b + 4ab^3 - 4abuv)Hk - 4a^2b^2 = 0.$

The principal curvature k_a and k_b are roots of the above equation.

$$\text{mean curvature } \mu = \frac{GL + EL - 2FM}{2(EG-F^2)}.$$

$$\mu = \frac{-2(a^2-b^2+uv)\left(\frac{-2a^2b^2}{H}\right)}{2H^2}$$

$$\mu = \frac{2a^2b^2(a^2-b^2+uv)}{H^3}$$

$$\text{Gaussian curvature } K = \frac{LN-M^2}{EG-F^2}$$

$$= \frac{-\left(\frac{-4a^2b^2}{H^2}\right)}{H^2}$$

$$= \frac{-4a^2b^2}{H^4}.$$

Theorem:- $\vec{r} = (a\sin u\cos v, a\sin u\sin v, a\cos u)$.

Show that all points on a sphere are umbilic.

Sol.

The representation of a point on a sphere with latitude u and longitude v as parameter is

$\vec{r} = (a\sin u\cos v, a\sin u\sin v, a\cos u)$ for the parametric representation of a point on a sphere we shall find first & second fundamental forms at a point and show that $L/E = M/F = N$. and so that every point on a sphere is umbilics.

$r_1 = (a\cos u\cos v, a\cos u\sin v, -a\sin u)$.

$r_{12} =$

$r_2 = (-a\sin u\sin v, a\sin u\cos v, 0)$.

$-a\cos u\sin v, a\cos u\cos v, 0$

$E = r_1 \cdot r_1 = a^2$

$F = r_1 \cdot r_2 = 0$

$G = r_2 \cdot r_2 = a^2\sin^2 u$.

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0. \end{vmatrix}$$

$= \vec{i}\,(a^2\sin^2 u\cos v) - \vec{j}\,(-a^2\sin^2 u\sin v)$

$+ \vec{k}\,(a^2\sin u\cos^2 v\cos u + a^2\sin u\cos u\sin^2 v)$.

$= (a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u)$.

$H = \sqrt{EG - F^2} = \sqrt{a^2 \sin^2 u - 0} = a^2 \sin u.$

Now, we shall use the scalar triple product to find L, M, N.

$LH = [r_{11}, r_1, r_2].$

Now, $[r_{11}, r_1, r_2] = r_{11} \cdot (r_1 \times r_2).$

$= (-a\sin u \cos v, -a \sin u \sin v, -a\cos u)$

$(a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u).$

$-a^3 \sin u$
$M = 0$
$N = a^3 \sin^3 u$

$= -a^3 \sin^3 u \cos^2 v - a^3 \sin^3 u \sin^2 v - a^3 \sin u \cos^2 u.$

$= -a^3 \sin^3 u (\cos^2 v + \sin^2 v) - a^3 \sin u \cos^2 u.$

$= -a^3 \sin u (\sin^2 u + \cos^2 u)$

$= -a^3 \sin u.$

Then, $L = \dfrac{[r_{11}, r_1, r_2]}{H} = \dfrac{-a^3 \sin u}{a^2 \sin u} = -a.$

$L = [r_{11}, r_1, r_2]$

$M = \dfrac{[r_{12}, r_1, r_2]}{H} = 0$

$N = \dfrac{[r_{22}, r_1, r_2]}{H} = \dfrac{-a^3 \sin^3 u}{a^2 \sin u} = -a\sin^2 u.$

$\dfrac{L}{E} = \dfrac{M}{F} = \dfrac{N}{G}$ gives $\dfrac{-a}{a^2} = \dfrac{0}{0} = \dfrac{-a^2 \sin^2 u}{a^2 \sin^2 u} = \dfrac{-1}{a}.$

Thus all points on a spheres are umbilics //

Second fundamental theorem.

Defn:.

* 2^{nd} order magnitude and 2^{nd} fundamental form.

The 2^{nd} derivative of $\bar{r}$ w.r to u, v at any point on the surface are denoted by,

$$\bar{r}_{11} = \frac{\partial^2 r}{\partial u^2}, \quad \bar{r}_{12} = \frac{\partial^2 \bar{r}}{\partial u \partial v}$$

$$\bar{r}_{21} = \frac{\partial^2 \bar{r}}{\partial v \partial u} \quad \bar{r}_{22} = \frac{\partial^2 r}{\partial v^2}.$$

The fundamental magnitude of 2^{nd} order an resolved parts of $\bar{r}_{11}, \bar{r}_{12}, \bar{r}_{22}$ in the direction of the normal to the surface of the point (u,v) and they are denoted by L, M, N.

Then, $L = \bar{r}_{11} \cdot \bar{N}$

$M = \bar{r}_{12} \cdot \bar{N}$

$N = \bar{r}_{22} \cdot \bar{N}$.

The quadratic $Ldu^2 + 2Mdvdu + Ndv^2$ is called the 2^{nd} fundamental form and L, M, N are called 2^{nd} fundamental magnitud

Normal curvature.

Defn:.

Let $\bar{r} = r(u,v)$ be a surface and $p(u,v)$ be points on it and $\bar{r} = \bar{r}(s)$ is a curve through p.

The normal curvature vector $\vec{r}''$ along the normal to the surface of p and is denoted by $\vec{k_n}$.

i.e) $\vec{k_n} = N r'' = N k \vec{n} = k(N \cdot \vec{n}) = k\cos\theta$.

If $\theta = 0$ then $\vec{k_n} = k$.

Meusnier's theorem.

If $\vec{k_n}$ denote the curvature of oblique and normal sections through the same tangent.

The angle between the principal normal $\vec{n}$ to a curve on the surface and the surface normal N is $k\cos\theta = \vec{k_n}$.

This is known as meusnier's theorem.

Alternate form for normal curvature.

Let $\bar{r} = \bar{r}(u,v)$ be a surface and $\bar{r} = \vec{k_n}$ be a curve on it. then u, v are friction of s for the point on the curve.

$$\vec{r}' = \frac{dr}{ds} = \frac{\partial \bar{r}}{\partial u} \cdot \frac{du}{ds} + \frac{\partial \bar{r}}{\partial v} \cdot \frac{dv}{ds}.$$

$$= \bar{r_1} \cdot u' + \bar{r_2}\, v'.$$

$$\vec{r}'' = \bar{r_1}\, u'' + \frac{d\bar{r_1}}{ds}\, u' + \vec{r_2}\, v'' + \frac{d\bar{r_2}}{ds} \cdot v'.$$

$\frac{\partial \bar{r}}{\partial u}$.

smallu

$$= \bar{r_1}\, u'' + \left(\frac{\partial r_1}{\partial u} \cdot \frac{du}{ds} + \frac{\partial r_1}{\partial v} \cdot \frac{dv}{ds} \right) u' + \bar{r_2}\, v''$$

$$r_{11}\, u'^2 \quad r_{21} v' u'$$

$$+ \left(\frac{\partial r_2}{\partial u} \cdot \frac{du}{ds} + \frac{\partial \bar{r_2}}{\partial v} \cdot \frac{dv}{ds} \right) v'.$$

$$r_{12} \quad u'v' + r_{22}\, v'^2$$

$$= \bar{r_1}\, u'' + \bar{r_{11}}\, u'^2 + \bar{r_{12}}\, u'v' + \bar{r_2}\, v'' + \bar{r_{21}} u'v'$$

$$+ \bar{r_{22}}\, v'^2.$$

$$\vec{r}'' = \bar{r_1} \cdot u'' + \bar{r_{11}}\, u'^2 + 2 \bar{r}_{12}\, u'v' + \bar{r_2}\, v'' + \bar{r_{22}} v'^2.$$

also $\vec{r}'' = k\vec{n}\, N + \lambda \vec{r_1} + \mu \vec{r_2}$.

$$\Rightarrow k\vec{n} = \vec{r}'' \cdot \bar{N}.$$

$$= (\bar{r}_{11} \bar{N})\, u'^2 + 2 (\bar{r}_{12} \bar{N})\, u'v' + (\bar{r_{22}} \bar{N})\, v'^2.$$

$$(\because \bar{r_1}\, \bar{N} = 0$$

$$= L \left(\frac{du}{ds} \right)^2 + 2M \left(\frac{du}{ds} \right) \left(\frac{dv}{ds} \right) + N \left(\frac{dv}{ds} \right)^2$$

$$= \frac{L du^2 + 2M du dv + N dv^2}{ds^2}$$

$$= \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}.$$

Types of points.

eliptic point:

If $LN - M^2 > 0$ then p is called an eliptic

Parabole point:-

If $LN-M^2=0$, then p is called parabole point.

hyperpole point:-

If $LN-M^2<0$, then p is called a hyperpole points.

Principal curvature:

Defn:-

The curvature of the principal direction of a surface through a given point on it are called the principal curvature of the point $ thus the principal curvature at a point on the surface is maximum & minimum curvature at a point.

Find the principal directions are at right angles.

proof:

we know that the normal curvature $k\vec{n}$. given direction (du, dv) through the point p(u,v). The surface is given by.

$$(Ll+Mm)+\lambda(El+Fm)=0 \longrightarrow (1).$$

$$(Ml+Nm)+\lambda(Fl+Gm)=0 \longrightarrow (2).$$

Eliminate r between ① and ②, we get.

$(Ll+Mm)(Fl+Gm) = (ml+Nm)(El+Fm)$.

$\Rightarrow (EM-FL)l^2 + (EN-GL)lm + (FN-GM)m^2 = 0$

Replace l, m by du, dv we get.

$(EM-FL)du^2 + (EN-GL)du\,dv + (FN-GM)dv^2 = 0$

Apply the condition of orthogonaly.

$(Pdu^2 + 2Q\,du\,dv + Rdv^2) = 0$,

$P = EM-FL \quad 2Q = EN-GL, \quad R = FN-GM$.

we know that.

$\frac{Edu^2}{R} - \frac{2F\,du\,dv}{Q} + \frac{G\,dv^2}{P}$

condition of $ER - 2FQ + GP = 0$.

we get.

$E(FN-GM) - F(EN-GL) + G(EM-FL)$,

identically,

corollary:-

The above quadratic fails to determine the principal directions when its coefficient variable identically

i.e) $EM-FL = EN-GL = FN-GM = 0$.

$\Rightarrow \frac{E}{L} = \frac{F}{M} = \frac{G}{N}$. Then P is called $\boxed{\text{umbilic}}$.

Lines of curvature.

Defn:-

A curve on a surface whose tangent at each point is along a principal direction is called lines of curvature.

Rodrigue's formula.

The necessary and sufficient condition that a curve on a surface to be a line of curvature is that $\frac{d\bar{N}}{ds} \propto \frac{d\bar{r}}{ds}$ or $d\bar{N} + Rd\bar{r} = 0$ at each of its point.

proof:-

Let (du, dv) be a line of curvature on the surface.

$\Rightarrow (du, dv)$ is a principal direction at (u, v) to the surface.

$\Rightarrow (L - KE)du + (M - KF)dv = 0 \longrightarrow ①$.

$(M - KF)du + (N - KG)dv = 0 \longrightarrow ②$.

where K is the one of the principal curvature we have,

$E = \bar{r_1}^2 \quad F = \bar{r_1} \cdot r_2 \quad G = \bar{r_2}^2$.

$L = r_{11} \cdot N$

$M = r_{12} \cdot N$

$N = r_{22} \cdot N$

$L = -\bar{N_1}\bar{r_1} \quad M = -\bar{N_1}\bar{r_2} \quad N = -\bar{N_2}\bar{r_2}$.

$① \Rightarrow (-N_1\bar{r_1} - K\vec{r_1}^2)du - (N_2\bar{r_1} - K\bar{r_1}\bar{r_2})dv = 0$.

$(N_1 du + N_2 dv)\bar{r_1} + K(\bar{r_1}du + \bar{r_2}dv)\bar{r_1} = 0$.

$\Rightarrow d\bar{N}\bar{r_1} + (Kd\bar{r}) \cdot \bar{r_1} = 0 \longrightarrow ③$.

$\Rightarrow (d\bar{N} + Kd\bar{r})\bar{r_1} = 0 \longrightarrow ③$

since $||^{ly}$ $② \Rightarrow (d\bar{N} + Kd\bar{r}) \cdot \bar{r_2} = 0 \longrightarrow ④$.

since $\bar{N} \cdot \bar{N} = 1 \Rightarrow \bar{N}^2 = 1$.

diff w.r to N we get.

$2\bar{N} \cdot d\bar{N} = 0.$

$\bar{N} \cdot d\bar{N} = 0.$

$\Rightarrow d\bar{N}$ is normal to $\bar{N}$.

$\Rightarrow d\bar{N}$ is a tangent vector.

$k d\bar{r} + d\bar{N}$ is a tangent vector.

also, r_1, r_2 are tangent vector.

so ③, ④ are possible only if.

$d\bar{N} + k d\bar{r} = 0 \Rightarrow \frac{d\bar{N}}{ds} \alpha \frac{d\bar{r}}{ds}$

$\Rightarrow d\bar{N} + k d\bar{r} = 0.$

Then by rotating the alone steps we find the direction at a point (u,v) are the curve satisfy ① & ②.

$\Rightarrow$ curve must be a line of curvature.

conversely,

assume that $d\bar{N} + k d\bar{r} = 0 \Rightarrow$.

curve must be a line of curvature.

Theorem

The necessary and sufficient condition the parametric curve be line of curvature are $F = 0$ and $M = 0$.

proof:-

Parametric curves are u = constant and v = constant.

$\Rightarrow du = 0$, $dv = 0 \Rightarrow du\,dv = 0 \longrightarrow$ ①

If the parametric curves are line of curvature then they must be orthogonal.

$\Rightarrow F=0$

The differential equation of line of curvature is $EM-FL=0$, $FN-GM=0$

$EN-GL\neq 0$.

$\Rightarrow EM=0$, $GM=0$. since $F=0 \Rightarrow M=0$.

Hence $F=0$, $M=0$ are necessary condition the parametric curve to be line of curvature.

conversely,

if $F=0$, $M=0$, the equation ② reduces to $(EN-GL)\,du\,dv=0$.

$\Rightarrow du\,dv=0 \quad (\because EN-GL\neq 0)$.

Hence condition of sufficient.

Theorem:-

Euler's formula.

The normal curvature $k\vec{n}$ at a point of a surface is given in terms ka, kb by the formula $k\vec{n}=ka\cos^2\psi + kb\sin^2\psi$.

where ψ is the angle between (du, dv) and $dv=0$.

proof

Let the line of curvature between as parametric curves then $F=0, M=0$.

Therefore $$k_n = \frac{L\,du^2 + N\,dv^2}{E\,du^2 + G\,dv^2} \longrightarrow (1)$$

Since the direction $u=$constant and $v=$constant are principal directions so that the curves for these directions are k_a, k_b respectively.

$$k_a = \frac{L}{E}, \quad k_b = \frac{N}{G}.$$

Since (du, dv) makes ψ with $v=$constant.

We have $$\cos\psi = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$$

$$= \sqrt{E}\,\frac{du}{ds} \quad (\because F=0).$$

$$\sin\psi = \frac{H}{\sqrt{E}}\cdot\frac{dv}{ds} = \frac{\sqrt{EG-F^2}}{\sqrt{E}}\,\frac{dv}{ds}$$

$$= \sqrt{G}\cdot\frac{dv}{ds} \quad (\because F=0).$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$
$$= E\,du^2 + G\,dv^2.$$

from for (1),

$$k_n = \frac{L\,du^2 + N\,dv^2}{ds^2}$$

$$= L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin^2\psi.$$

$\kappa_n = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi$

Theorem.

A necessary and sufficient condition for a surface to be a developable is that it's Gaussian curvature shall be zero.

proof:

If a developable is a cylinder (or) cone the Gaussian curvature is evidently.

Take the edge of regression as

$R = \bar{r} + u\bar{t}$ where r, u are parameters.

$$\bar{R_1} = \frac{\partial \bar{r}}{\partial s} + u\frac{d\bar{t}}{ds} = \bar{t} + u\bar{t}' = \bar{t} + uk\bar{n}$$

$(1+uk)(1)$

$$\bar{R_2} = \frac{\partial \bar{r}}{\partial u} + \vec{t} = 0 + \bar{t},$$

since $\bar{r} = \bar{r}(s) = \vec{t}$.

$$\bar{R_{11}} = \bar{t}' + uk'\vec{n} + uk\vec{n}' = k\bar{n} + uk'\bar{n} + uk(\tau b - k\bar{t})$$

$$= k\bar{n} + uk'\bar{n} + uk\tau\vec{b} - uk^2\bar{t}$$

$$= (-uk^2)\bar{t} + (k + uk')\vec{n} + uk\tau\vec{b}.$$

$\bar{R_{12}} = \bar{R_{21}} = 0$

$\bar{R_{22}} = 0;\ E = \bar{R_1}^2 = 1 + u^2k^2$

$F = \bar{R_1}\bar{R_2} = 1 \cdot G = \bar{R_2}^2 = 1,$

$H^2 = EG - F^2 = u^2k^2$

$H = uk$

$HL = [R_1, R_2, R_{11}] = (R_1 \times R_2) \cdot R_{11}$

$= [(\bar{E} + uk\dot{\bar{n}}) \times \bar{E}] \cdot R_{11}$

$= 0 + (uk\bar{b}) \cdot R_{11}.$

$= -ukb \cdot [uk\tau\bar{b}].$

$HL = -u^2k^2\tau$

$HM = [R_1, R_2, R_{12}] = (R_1 \times R_2) \cdot R_{12}$

$= -uk\bar{b} \cdot 0$

$HM = 0.$

$HN = [R_1, R_2, R_{22}] = (R_1 \times R_2) \cdot R_{22}$

$HN = 0.$

Gaussian curvature $K = \frac{LN - M^2}{H^2} = 0.$

since $M = 0$, $N = 0$ hence $K = 0$ is necessary.

conversely let $K = 0$.

$\Rightarrow \frac{LN - M^2}{H^2} = 0 \Rightarrow LN - M^2 = 0 \longrightarrow ①$

we have,

$L = -\bar{N}_1 \bar{r}_1 \quad M = \bar{N}_1 \cdot \bar{r}_2 = -\bar{N}_2 \cdot \bar{r}_1$

$N = -\bar{N}_2 \cdot \bar{r}_2.$

$① \Rightarrow (-\bar{N}_1 \cdot \bar{r}_1)(-\bar{N}_2 \cdot \bar{r}_2) - (\bar{N}_1 \cdot \bar{r}_2)(\bar{N}_2 \cdot \bar{r}_1) = 0$

$\Rightarrow (r_1 \times r_2)(N_1 \times N_2) = 0$

$\Rightarrow H\bar{N}(\bar{N}_1 \times \bar{N}_2) = 0 \Rightarrow H[\bar{N}, \bar{N}_1, \bar{N}_2] = 0$

$[\bar{N}, \bar{N}_1, \bar{N}_2] = 0$

(i) $\bar{N}, \bar{N}_1, \bar{N}_2$ are coplanes (or).

(i) $N_1 = 0$ (or) $N_2 = 0$ (or)

(ii) $N_1 = \lambda N_2$.

case (i):-

$\bar{N} \cdot \bar{N_1} = 0$ & $\bar{N} \cdot \bar{N_2} = 0$.

$\Rightarrow \bar{N} = \mu(N_1 \times \bar{N_2})$.

$\bar{N}, \bar{N_1}, \bar{N_2}$ are not coplaner.

Hence (i) is not possible $(\bar{N_1}) = 0$).

case (ii)

eqn of tangent plane at $p(\bar{r})$ is $(R-\bar{r})\bar{N} = 0$.

$\Rightarrow [(R-\bar{r}) \cdot \bar{N}] = 0$

$\Rightarrow [(\bar{R}-\bar{r})\bar{N_1} - \bar{r_1} \cdot N)$

$\Rightarrow \bar{r_1} \cdot N = 0 \Rightarrow$ surface is developable.

$|||^{ly}$ for $\bar{N_2} = 0$.

case (iii).

$$\bar{N_1} = \lambda \bar{N_2}.$$

here we apply the proper parameter.

$u = u' + v' \quad v = u' - \lambda v'$.

Here $\frac{\partial N}{\partial u'} = N_1' = \frac{\partial N}{\partial u} \cdot \frac{\partial u}{\partial u'} + \frac{\partial N}{\partial v} \cdot \frac{\partial v}{\partial u'} = N_1 + N_2$.

$$\frac{\partial N}{\partial v'} = N_2' = \frac{\partial N}{\partial u} \cdot \frac{\partial u}{\partial v'} + \frac{\partial N}{\partial v} \cdot \frac{\partial v}{\partial v'} = N_1 + N_2.$$

$\Rightarrow$ tangent planes involves one parameter.

hence the surface is developable.

hence the condition is sufficient.

Hence the condition associated with curves on Developable surface (s) theorem.

(x) monge theorem:-

characterings line of curve

on a surface.

Let $r = r(u,v)$ be a surface.

$\bar{r} = r(s)$ be a curve on it. N denotes a normals to the surface of the point $p(\bar{r})$.

Now consider the surface formed by the normals to the given surfaces.

$\bar{r} = \bar{r}(u,v)$ along the curve $\bar{r} = \bar{r}(s)$ Any on this surface will have position vector.

$\bar{R} = \bar{r}(s) + wN(s)$ (or) $\bar{R} = \bar{r} + w\bar{N}$ $\longrightarrow$ ①

where s and w are the two parameters on the surface.

Then $\frac{\partial \bar{R}}{\partial s} = R_1 = \frac{\partial \bar{r}}{\partial s} + w\frac{\partial \bar{N}}{\partial s} = \bar{r} + w\bar{N}'$.

$$\frac{\partial \bar{R}}{\partial s} = R_2 = \bar{N}$$

Let the unit normal to the surface at $p(r)$ denoted by N^*.

Then $N^* = \bar{t} \times \bar{N}$

$\therefore R_{12} = R_{21} = N'$ & $R_{22} = 0$.

The surface is a developable iff its Gaussian curvature $K = 0$

$$\Rightarrow \frac{LN - M^2}{EG - F^2} = 0 \Rightarrow M = 0 \Rightarrow [E, N, N'] = 0.$$

Hence the surface normal along the curve from a developable iff $[E, N, N'] = 0$

Now, it remains to the proved that

$[E, N, N'] = 0$ is the necessary and sufficient condition for the curve to by a line of curvature.

$[E, N, N'] = 0 \Rightarrow \vec{E} \times N' = 0.$

$\Rightarrow N' = -\varepsilon t$ for some scalars $\Rightarrow \frac{dN}{ds} + \varepsilon \frac{dr}{ds} = 0.$

$dN + \varepsilon dr = 0.$

which is the Rodrigue's formula.

$\Rightarrow$ given curve is a line a curvature.

conversely,

If the eqn curve is a line of. curvature then by Rodrigue's formula.

$dN + \varepsilon d\bar{r} = 0 \Rightarrow N' = -\varepsilon t \rightarrow N' \times \vec{E} = 0.$

$\Rightarrow [E, N, N'] = 0.$

ϕ is angle between the principal norm to a curve on a surface

Hence the theorem //

$k_n = k\cos\phi$

$k_n = k\cos\phi$

Meusnier's theorem: (pri no, surfac and

If ϕ is the angle between the principal normal to a curve on a surface. and the surface normal N. then,

$k_n = k\cos\phi.$

proof:

we know that, At any point on a curve on a surface.

$\bar{r}'' = k_n N + \lambda r_1 + \mu r_2 \longrightarrow ①.$

taking the dot product with N on both sides.

$r''. N = k_n . N.N + \lambda r_1 . N + \mu r_2 . N.$

$[\because N.r_1 = N.r_2 = 0].$

$\Rightarrow r''. N = k_n.$

$\Rightarrow k\vec{n} = r''. N \longrightarrow ②.$

But we know that, $r'' = \frac{dt}{ds} = kn \longrightarrow ③$

sub ③ in ②.

$\Rightarrow k_n = k n N$

$\Rightarrow k_n = k \cos\phi \qquad [\because n.N = |n||N| \cos\phi$

$= \cos\phi.$

Thus the normal curvature k_n is the projection on the surface normal N of the vector of length k along the principal normal to the curve.

find the second fundamental form for the general surface resolution.

Sol.

wkt,

$r = [g(u)\cos v, g(u)\sin v, f(u)]$

$r_1 = [g_1(u)\cos v, g_1(u)\sin v\ f_1(u)]$

$r_2 = [-g(u)\sin v, g(u)\cos v, 0]$

$E = r_1^2 = g_1^2 + f_1^2.$

$F = r_1 \cdot r_2 = 0$

$G = r_2^2 = g^2 \quad H = \sqrt{EG - F^2} = g\sqrt{g_1^2 + f_1^2}.$

$$r_1 \times r_2 = \begin{vmatrix} \vec{E} & \vec{n} & \vec{b} \\ g_1(u)\cos v & g_1(u)\sin v & f_1(u) \\ -g(u)\sin v & g(u)\cos v & 0. \end{vmatrix}$$

$$= \vec{E}\,(0 - f_1 g \cos v) - \vec{n}\,(0 + f_1 g \sin v) +$$

$$\vec{b}\,(g g_1 \cos^2 v + g g_1 \sin^2 v).$$

$$= (-g f_1 \cos v,\ - f_1 g \sin v,\ g g_1).$$

$r_{11} = [g_{11}(u)\cos v,\ g_{11}(u)\sin v,\ f_{11}(u)]$

$r_{22} = [-g(u)\cos v,\ -g(u)\sin v,\ 0].$

$r_{21} = (-g_1(u)\sin v,\ g_1(u)\cos v,\ 0) = r_{12}$

$LH = r_{11}(r_1 \times r_2).$

$$= (-g_{11} g f_1 \cos^2 v - g_{11} f_1 \sin^2 v + f_{11} g g_1)$$

$$= -g_{11} g f_1 (\cos^2 v + \sin^2 v) + f_{11} g g_1$$

$$= -g_{11} g f_1 + f_{11} g g_1$$

$$= g(-f_1 g_{11} + f_{11} g_1)$$

$$L = \frac{g(-f_1 g_{11} + f_{11} g_1)}{g\sqrt{g_1^2 + f_1^2}}$$

$MH = r_{21}(r_1 r_2) = r_{12}(r_1 \times r_2)$

$= gg_1 f_1 \cos v \sin v - gg_1 f_1 \cos v \sin v.$

$MH = 0$

$M = 0$

$NH = r_{22}(r_1 \times r_2)$

$= f_1 g^2 \cos^2 v + g^2 f_1 \sin^2 v + 0.$

$NH = g^2 f_1 (\cos^2 v + \sin^2 v)$

$$N = \frac{g^2 f_1}{g\sqrt{g_1^2 + f_1^2}}$$

$$N = \frac{g f_1}{\sqrt{g_1^2 + f_1^2}}$$

$$II = \frac{-1}{\sqrt{g_1^2+f_1^2}} [g_{11}(u) f_1(u) - f_{11}(u) g_1(u)) du^2 - g(u) f_1(u) dv^2].$$

where II denote the second fundamental form.

find the normal curvature of the right helicoid $r(u,v) = (u\cos v, u\sin v, cv)$ at point on it.

proof:

$r = (u\cos v, u\sin v, cv)$

$r_1 = (\cos v, \sin v, 0)$

$r_2 = (-u\sin v, u\cos v, c).$

$E = r_1^2 = \cos^2 v + \sin^2 v = 1.$

$F = r_1 \cdot r_2 = -u \sin v \cos v + u \sin v \cos v = 0$

$G = r_2^2 = u^2 \sin^2 v + u^2 \cos^2 v + c^2$

$= u^2(\sin^2 v + \cos^2 v) + c^2$

$= u^2 + c^2.$

$H = \sqrt{EG - F^2}$

$= \sqrt{(u^2+c^2) - 0}$

$= \sqrt{u^2+c^2}.$

$N = \dfrac{r_1 \times r_2}{H}$

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & c \end{vmatrix}$$

$= \vec{i}(c\sin v - 0) - \vec{j}(c\cos v + 0) + \vec{k}$
$(u\cos^2 v + u\sin^2 v).$

$= \vec{i}\, c\sin v - j\, c\cos v + \vec{k}u$

$= (c\sin v, -c\cos v, u).$

$N = \dfrac{(c\sin v, -c\cos v, u)}{\sqrt{u^2+c^2}}$

$r_{11} = (0,0,0)$ $r_{12} = (-\sin v, \cos v, 0) = r_{21}.$

$r_{22} = (-u\cos v, -u\sin v, 0).$

Now, $L = r_{11} \cdot N = 0.$

$M = r_{12} \cdot N = (-\sin v, \cos v, 0) \dfrac{(c\sin v, -c\cos v, u)}{\sqrt{u^2+c^2}}.$

$$= \frac{-c\sin^2 v - c\cos^2 v + 0}{\sqrt{u^2+c^2}}$$

$$= \frac{-c(\sin^2 v + \cos^2 v)}{\sqrt{u^2+c^2}}$$

$$= \frac{-c}{\sqrt{u^2+c^2}}.$$

$N = r_{22} \cdot N$

$= (-u\cos v, -u\sin v, 0) \dfrac{(c\sin v, -c\cos v, u)}{\sqrt{u^2+c^2}}$

$$= \frac{-cu\cos v\sin v + cu\sin v\cos v}{\sqrt{u^2+c^2}}$$

$= 0.$

$$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

$$= \frac{-2c\,du\,dv}{\sqrt{u^2+c^2}\,(du^2 + (u^2+c^2)dv^2)}$$

Prove that ~~iff~~ L, M, N vanish at all points of a surface, then the surface is a plane.

Soln

Let us assume that the surface to be a plane surface, then the normal N is constant vector so that.

$N_1 = N_2 = 0 \longrightarrow (1)$

now,

$L = -N_1 r_1 \quad M = -N_1 \cdot r_2 = -N_2 r_1$

$N = N_2 \cdot r_2 \longrightarrow (2)$

eqn (1) & (2) we have.

$L = M = N = 0.$

conversely,

assume that $L = M = N = 0.$

Then normal N is a constant vector from. hypothesis we have.

$N_1 r_2 = N_2 r_1 = 0 \longrightarrow (3)$

eqn (3) & (4) we get.

N_1 & N_2 are $\parallel^{el}$ to N.

so let them be $N_1 = \lambda N$

$N_2 = \mu N.$

where λ and μ are constant.

since $N \cdot N = 1$ we have from the above.

$$\left.\begin{aligned} N_1 \cdot N &= \lambda N \cdot N \Rightarrow N_1 N = \lambda \\ N_2 \cdot N &= \mu N \cdot N \Rightarrow N_2 N = \mu \end{aligned}\right\} \longrightarrow (5)$$

further from $N \cdot N = 1.$

we get.

$$\left.\begin{array}{l} N \cdot N_1 = 0 \\ N \cdot N_2 = 0 \end{array}\right\} \longrightarrow (6)$$

eqn (6) & (5).

$\lambda = \mu = 0$ this proves that $N_1 = N_2 = 0$.

thus normal N is a constant vector.

The given surface is a plane surface.

Monge theorem:-

Statement:-

(7) A necessary and sufficient condition that a curvature curve on a surface be a line of curvature is that the surface normals of monge along the curve from a developable.

Monge theorem

Proof:

To prove the theorem.

we consider the surface formed by a normals along the curve $\vec{r} = \vec{r}(s)$ any point on the surface will have position vector $\vec{R} = \vec{r}(s) + v\vec{N}(s)$.

Here s and v are parameters simply

we write.

$$\vec{R} = \vec{r} + v\vec{N}$$

$$\vec{R_1} = \vec{t} + v\vec{N}'$$

$$\vec{R_2} = \vec{N}$$

$$\vec{R_{12}} = \vec{N}' = \vec{R_{12}}$$

$$\vec{R_{22}} = 0$$

But

$[\vec{R_1}, \vec{R_2}, \vec{R_{12}}] = HM$ and $[\vec{R_1}, \vec{R_2}, \vec{R_{22}}] = H\vec{N}$.

$$HM = [\vec{t} + v\vec{N}, \vec{N}, \vec{N}']$$

$$= [\vec{t}, \vec{N}, \vec{N}'] + [v\vec{N}, \vec{N}, \vec{N}']$$

$$HM = [\vec{t}, \vec{N}, \vec{N}']$$

$$HN = [\vec{t} + v\vec{N}, \vec{N}, 0]$$

$$= [\vec{t}, \vec{N}, 0] + [v\vec{N}, \vec{N}, 0]$$

$$HN = 0$$

$$\Rightarrow N = 0.$$

The Gaussian curvature is developable.

iff Gaussian curvature $K=0$

ie) iff $\frac{LN-M^2}{EG-F^2}=0$

$$LN-M^2=0$$

$$0-M^2=0 \quad [\text{since } N=0]$$

$$M^2=0$$

$$M=0$$

ie) iff $[\vec{t}, \vec{N}, \vec{N}']=0$

The surface normal along the curve from a developable iff $[\vec{t}, \vec{N}, \vec{N}']=0$.

It remains to prove that this condition is satisfied iff the curve is a line of curvature.

Since $\vec{t}\times\vec{N}'$ is normal to the given surface eqn.

$$[\vec{t}, \vec{N}, \vec{N}']=0$$

$$\Rightarrow \vec{t}\times\vec{N}'=0$$

ie) $\vec{N}'=-k\vec{t}$ for some fn k.

conversely,

If $\vec{N}'=-k\vec{t}$.

then scalar triple product.

$$[\vec{t}, \vec{N}, \vec{N}']=0.$$

But the eqn, $\vec{N}'=-k\vec{t}$ is a

$[t\,N\,N']=0$
$N'=\ldots$